

Exploring connectivity of random subgraphs of a graph.

Connectivity of random subgraphs of Cartesian products of K_2 , K_3 , and P_3 . A survey of
uniformly most reliable networks.

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Abstract

This work is divided into two main parts. The first part is devoted to exploring the connectivity of random subgraphs of cartesian products of K_1 , K_2 , and P_3 . In the second part, the author presents a short review of the results about network reliability.

The cartesian product of K_2 , the complete graph with 2 vertices, is the cube graph Q^n . A random subgraph of Q^n , $Q_{p_n}^n$, contains all vertices of Q^n , and each edge of Q^n independently with probability p_n . One can call p_n the percolation parameter. The author explains in detail that for $p_n \geq 1 - (1/2)(\log n)^{1/n}$, $Q_{p_n}^n$ has no components with size larger than 1 and smaller than 2^n , as $n \rightarrow \infty$. It is also explained that for $p_n = 1 - (1/2)\lambda^{1/n}(1 + o(1/n))$, the probability that $Q_{p_n}^n$ has no isolated point, as $n \rightarrow \infty$, tends to $e^{-\lambda}$; hence, the probability that $Q_{p_n}^n$ is connected tends to $e^{-\lambda}$. For constant percolation values larger than $1/2$, when n tends to infinity, almost every random subgraph of Q^n is connected; for percolation values smaller than $1/2$, when n tends to infinity, almost no random subgraph of Q^n is connected; and for percolation values equal $1/2$, when n tends to infinity, the probability that $Q_{1/2}^n$ is connected tends to e^{-1} . At the end of this section, a comparison between connectivity of a typical random graph with M edges and N vertices, $G \in G(N = 2^n, M = n2^{n-1})$, and Q^n , after percolation with the parameter p_n is presented.

This work continues with exploring the threshold function for the cartesian product of K_3 , the complete graph with 3 vertices, denoted by ${}^3Q^n$. It is shown that for $p_n \geq 1 - (1/\sqrt{3})(\log n)^{1/n}$, ${}^3Q_{p_n}^n$ has no components with size larger than 1 and smaller than 3^n , as $n \rightarrow \infty$. Then, it is proved that for $p_n = 1 - (1/\sqrt{3})\lambda^{1/2n}(1 + o(1/n))$, the probability that ${}^3Q_{p_n}^n$ has no isolated point, as $n \rightarrow \infty$, tends to $e^{-\lambda}$; hence, the probability that ${}^3Q_{p_n}^n$ is connected tends to $e^{-\lambda}$. At last, the author suggests that the threshold value for connectivity of the cartesian product of P_3 , where P_3 is a path with length 2, denoted by P_3^n , is $2 - \sqrt{2}$. One can show that for percolation values smaller than $2 - \sqrt{2}$, almost no random subgraph of P_3^n is connected, and for percolation values larger than $2 - \sqrt{2}$, almost every random subgraph of P_3^n has no isolated point. The author also shows that for percolation values larger than 0.68 almost all random subgraphs of P_3^n are connected.

The last part of this work, sheds light on reliability of networks. The main question in this part is: one is given 2 parameters, n and m where n and m are positive integers. Among all graphs with n vertices and m edges, which graph G , if any, maximizes the probability that when one does percolation on G with the parameter p_n , for all p_n in $(0, 1)$ there is one component? G would be called the uniformly optimally reliable graph (UOR graph) for the parameter n and m . It is shown in this part, for some m and n there is no UOR graph, since the graph which maximizes the probability of connectivity depends on p_n in that family of graphs. A review of results about when the UOR graph exists is presented in this part.

Keywords: Random subgraphs, percolation, n-cube, path graph, reliable networks.

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CHAPTER 1

INTRODUCTION

Exploring the connectivity of random subgraphs of different families of graphs is one of the most interesting topics in random graphs and percolation theory. A random subgraph of a graph $G(V_n, E_m)$ is a graph which contains all vertices of G , and each edge of G independently with probability p_n . p_n is known as the percolation parameter. Connectivity of a random subgraph of a graph can be investigated both for small and considerably large (when n tends to infinity) graphs. For considerably large graphs, the first step in exploring the connectivity is to calculate p_c which for all constant $p = p_n, p \in (0, 1)$ and $p < p_c$, as n tends to infinity, almost all random subgraphs of $G(V_n, E_m)$ is connected; but for all $p \in (0, 1)$ and $p > p_c$, as n tends to infinity, almost no random subgraphs of $G(V_n, E_m)$ is connected. The second step is to investigate what happens when $p = p_c$. A more complete approach is to calculate p_c when p_c depends on n .

For small graphs, it is of interest to find the uniformly optimally reliable graph (UOR graph). Consider $G(n, m)$ as the family of graphs with n vertices and m edges. The UOR graph is the graph $G \in G(n, m)$ that maximizes the probability that G is connected after percolation with the parameter p_n for fixed n, m and all $p_n \in (0, 1)$.

One of the interesting graphs for analyzing the connectivity of its random subgraphs is the cube graph. The cube graph Q^n , is a graph with the vertices labeling $1, 2, 3, 4, \dots, 2^n - 1$. Two vertices in this graph are adjacent if their binary representation differs only in one digit. Another way to define Q^n is using the cartesian products of n copies of K_2 , where K_n is the complete graph with n vertices. It is shown by Paul Erdős and Joel Spencer [4] that $p_c = 1/2$ for Q^n . An extension of Q^n is the graph with the vertices labeling $1, 2, 3, 4, \dots, 3^n - 1$, where two vertices in this graph are adjacent if their ternary representation differs only in one digit. We call this graph 3-cube denoted by ${}^3Q^n$. One can show that ${}^3Q^n$ is the cartesian product of K_3 . Connectivity of random subgraphs of the cartesian product of K_i is investigated by Lane Clark [15]. Another extension of Q^n is the cartesian product of n copies of a path with length 2 which we call it P_3^n .

This thesis is divided into two main parts. The first part (chapter 4,5,6) is devoted to the connectivity of random subgraphs of some considerably large graphs, and the second part is the connectivity of random subgraphs of small graphs (chapter 7). Chapter 2 presents a very short review of definition and results in graph theory. Chapter 3 is a short review of the definitions in random graph theory; it is explained briefly in this chapter that how small components construct a giant component and gradually a graph becomes connected by adding more edges to it. In chapter 4, the results by Bela Bollobás [1] on finding p_c for connectivity of random subgraphs of Q^n is explained in detail. In chapter 5 the author calculates p_c for connectivity of random subgraphs of ${}^3Q^n$. After calculating the threshold value for the connectivity of random subgraphs of ${}^3Q^n$, the author found that this problem is solved for a general case of the random subgraphs of cartesian product of K_i [15]. Chapter 6 is an approach to find p_c for connectivity

of random subgraphs of P_3^n . This work finishes with chapter 7 which is a review of the results on finding the UOR graph. In this chapter it is shown by the author that for some m and n there are no UOR graph.

CHAPTER 2

GRAPH THEORY BACKGROUND

This chapter presents a short review of basic definitions in graph theory. Most of the definitions in this chapter are extracted from [10].

2.1 Graph models and their matrix representation

Graphs

Definition 1. Graph: A graph $G(V, E)$ is an ordered pair consisting of the set of *vertices* V , and the set of *edges* E . Each edge is associated with a set of vertices which are called *endpoints*. Two vertices are *adjacent* if they are joined by an edge. Two edges are *adjacent* if they have a common endpoint. A vertex is *incident* to an edge and viceversa, if that vertex is an endpoint of the edge. A *self-loop* is an edge which joins a vertex to itself. A *multi-edge* is a set of two or more edges having the same endpoints. A *simple graph* is a graph without self-loops and multi-edges.

Degrees

After defining a graph, it is of interest to get familiar with the characteristics of different graphs in order to compare them. One of the basic characteristics is the degree of each vertices.

Definition 2. Degree: The *degree* of a vertex, denoted by $\deg(v)$, is the number of edges incident on that vertex plus two times the number of its self-loops. The *smallest degree* in a graph is denoted by δ_{min} or δ , and the *largest degree* in a graph is denoted by δ_{max} or Δ . The *degree sequence* of a graph is the non-increasing sequence of vertex degrees.

The first question that comes into mind, after defining the degree sequence of a graph, is if there exists a degree sequence of a graph for each sequence of positive integers.

Definition 3. Graphic: A sequence of positive integers is *graphic* if there is a permutation of it that is the degree sequence of a simple graph. An explicit sufficient and necessary condition for a sequence of positive integers to be graphical is:

Theorem 1. A sequence of non-negative integers (d_1, d_2, \dots, d_n) is graphical if and only if

$$\sum_{i=1}^k \deg(i) \leq k(k-1) + \sum_{j=k+1}^n \min(k, \deg(i)) \quad (2.1)$$

for each $1 \leq k \leq n$ [9].

Graph models

There are many types of graphs. Some of the most important types of simple graphs are:

Definition 4. Common families of graphs: A *complete graph* K_n is a simple graph with n vertices which every pair of vertices is connected by an edge. A *bipartite graph* G is a graph with the set of vertices that can be partitioned into two subsets U and W , such that each edge in G has one endpoint in U and one endpoint in W . A *regular graph* is a graph where all vertices have the same degree. A *path graph* is a simple graph with $|V| = |E| + 1$ that can be drawn, such that all vertices and edges are in a single straight line. A *path graph* with $|V| = n$ and $|E| = n - 1$ is denoted by P_n . A *hypercube graph* (*cube graph*) is a simple n -regular graph with the set of vertices labels from 0 to $2^n - 1$, in which two vertices are adjacent if their binary representation differs only in one digit.

Definition 5. Subgraphs: H *subgraph* of G is a graph whose vertices and edges are in G . If $V_G = V_H$ then the subgraph H is said to *span* the graph G . The *induced subgraph* on $U \subseteq V_G$ of G is the graph whose set of vertices is U , and set of edges is all edges of G with two endpoints in U . The *induced subgraph* on $D \subseteq E_G$ of G is the graph whose its set of edges is D , and its set of vertices is all vertices which are incident with an edge in D . A maximal connected subgraph of a graph G is a *component* of G .

Definition 6. Cartesian product of a graph: $G \times H$, the *Cartesian product* of G and H is the graph with the set of vertices $V_G \times V_H$ and the set of edges $(V_G \times E_H) \cup (E_G \times V_H)$.

Defining a walk on a graph can help us to define some important characteristic of a graph such as connectivity of a graph and spanning trees.

Definition 7. Walk: In a graph G , a *walk* from vertex v_0 to vertex v_n is an ordered sequence

$$W = \langle v_0, e_1, v_1, e_2, \dots, v_{n-1}, e_n, v_n \rangle \quad (2.2)$$

of vertices and edges, such that the endpoints of e_i is $\{v_{i-1}, v_i\}$ for $i = 1, \dots, n$. For a simple graph one can abbreviate the representation as a vertex sequence

$$W = \langle v_0, v_1, \dots, v_n \rangle \quad (2.3)$$

Definition 8. Tree, spanning tree: A *path* is a walk with no repeated vertices (except the initial and final vertices). A *cycle* is a nontrivial closed path. A *tree* is a connected graph without cycle. A spanning tree of a graph is a subgraph of a graph which is a tree.

Matrix representations

The last important concept in this section is that each graph can be presented as a matrix, as follows:

Definition 9. Matrix representation of a graph: The *adjacency matrix* of a simple graph G is: $A_G[u, v] = \begin{cases} 1, & \text{if } u \text{ and } v \text{ are adjacent;} \\ 0, & \text{otherwise.} \end{cases}$ for all pairs of vertices u and v in V_G .

2.2 Connectivity

Connectivity of a graph is one of the most important property of a graph. A graph is *connected* if for every pair of vertices u and v , there is walk from u to v . There are different types of connectivity for a graph:

Definition 10. Vertex-connectivity: $\kappa_v(G)$, *vertex connectivity* of a connected graph G , is the minimum number of vertices which its removal will disconnect G or reduce it to a single vertex graph. A graph G is *k-connected* if $\kappa_v(G) \geq k$.

Definition 11. Edge-connectivity: $\kappa_e(G)$, *edge connectivity* of a connected graph G , is the minimum number of edges which its removal will disconnect G . A graph G is *k-edge-connected* if $\kappa_e(G) \geq k$.

Definition 12. Algebraic connectivity: The *Laplacian matrix* of a graph is $L := D - A$, where A is the adjacency matrix and the D is the diagonal matrix of vertex outdegrees. The algebraic connectivity of an undirected graph with the Laplacian matrix L is the second smallest eigenvalue of L . If one arranges the eigenvalues of L as : $\lambda_1(L) \leq \lambda_2(L) \leq \dots \lambda_n(L)$, then $\lambda_2(L)$ is the algebraic connectivity of a graph. The following theorem presents some applications of algebraic connectivity:

Theorem 2. For an undirected graph with minimum vertex degree δ and maximum vertex degree Δ , we have:

- $\lambda_2 \geq 0$ with the inequality strict if and only if the graph is connected.
- $\lambda_2 \leq \frac{n}{n-1}\delta \leq \frac{n}{n-1}\Delta \leq \lambda_n$.

CHAPTER 3

RANDOM GRAPH THEORY

BACKGROUND

A *random graph* is a graph with a specific number of vertices which adjacency between two vertices are determined in a random way [10]. In this chapter, we define Erdős Rényi random graph, and then we explain briefly some properties of it.

3.1 Evolution of Erdős Rényi graphs

$G(n, M)$, $0 \leq M \leq \binom{n}{2}$, is the equiprobable space of all simple graphs with the vertex set $V = \{1, 2, \dots, n\}$ and M edges. $G(n, p)$, $0 < p < 1$, is the collection of all graphs with the vertex set $V = \{1, 2, \dots, n\}$ in which two vertices are connected independently with the probability p [10], [1].

It is of interest to study the global structure of a random graph of order n (with n vertices) and size $M(n)$ (with $M(n)$ edges). Let us define $L_j(G)$ as the order of the j th largest component of a graph G , where if G has fewer than j components then $L_j(G) = 0$. Consider the random graph process $\tilde{G} = (G_t)_{t=0}^N$ where G_t is getting larger by adding more and more edges. When $t \sim \frac{1}{2}cn$ and $c < 1$ then in a.e G_t the maximum of the order of its components is of order $\log n$. When $c = 1$, in a.e G_t $L_1(G_{\lfloor n/2 \rfloor})$ has order $n^{2/3}$. When t passes $n/2$, $L_1(G)$ begins to grow suddenly and the giant component, which is a component whose order is much larger than other components, appears. Eventually, small components join the giant component and the graph becomes connected. Erdős Rényi proved that $(n/2) \log n$ is the sharp threshold for connectedness [1]. The following theorem illustrates this fundamental result:

Theorem 3. Let $c \in \mathbb{R}$ be fixed and let $M = (n/2)\{\log n + c + o(1)\} \in \mathbb{N}$ and $p = \{\log n + c + o(1)\}/n$. Then [1]:

$$\mathbf{P}(G_M \text{ is connected}) \rightarrow e^{-e^{-c}} \text{ as } n \rightarrow \infty \quad (3.1)$$

and

$$\mathbf{P}(G_p \text{ is connected}) \rightarrow e^{-e^{-c}} \text{ as } n \rightarrow \infty. \quad (3.2)$$

For more information regarding random graphs one can check [1], [2], [13], [14].

3.2 Properties of almost every graphs

A graph property T is true for almost every (all) graph if for fixed $p = p(n)$ [3]:

$$\lim_{n \rightarrow \infty} \mathbf{P}(G \in G(n, p) \text{ and } G \text{ has property } T) = 1 \text{ for } p > 0 \quad (3.3)$$

Some of the important "almost every graph properties" are:

Theorem 4. For any integer $r \geq 1$ and all $p \in (0, 1)$, almost every graph contains K_r [10].

Theorem 5. Almost every graph is connected for all $p \in (0, 1)$ [10].

Theorem 6. For $k \in \mathbb{N}$ and all $p \in (0, 1)$, almost every graph is k -connected [10].

3.3 Probabilistic methods

Usually, the goal in the probabilistic method is to prove the existence of a combinatorial structure with a certain property. The usual approach in these methods is to first construct a suitable probability space, then show that there exists a random object in that space with the desired properties [11]. Sometimes it is not easy to find the desired object, instead one proves that there is an object which *almost* satisfies the desired conditions [12]. Usually, it is possible to modify the almost close object in a deterministic way so that one gets the desired object. Markov's inequality, and Chebyshev inequality are two important inequalities used for this purpose. An important concept in probabilistic methods is the definition of threshold function, which is:

Definition 13. $r(n)$ is called a threshold function for a graph property T for $G(n, M(n))$ if:

1. When $\lim_{n \rightarrow \infty} \frac{M(n)}{r(n)} = 0$ almost every graphs do not satisfy T .
2. When $\lim_{n \rightarrow \infty} \frac{M(n)}{r(n)} = 1$ almost every graphs satisfy T .

CHAPTER 4

CONNECTED RANDOM SUBGRAPHS OF THE CUBE

1

The cube graph, Q^n , is a graph with 2^n vertices. If one labels each vertex of Q^n from 0 to $2^n - 1$, then two vertices are adjacent if their binary representation differs only in one digit. Hence, one can conclude that each vertex in Q^n is connected to n other vertices. In other words, Q^n has $n2^{n-1}$ edges. A random subgraph of Q^n is denoted by $Q_{p_n}^n$. $Q_{p_n}^n$ contains all vertices of Q^n , and each edge of Q^n independently with probability p_n .

It is of interest in this chapter to explore a critical value p_c , which for fixed values of p if $p < p_c$ then the probability that $Q_{p_n}^n$ is connected, as $n \rightarrow \infty$, tends to 0; but if $p > p_c$ then the probability that $Q_{p_n}^n$ is connected, as $n \rightarrow \infty$, tends to 1. Burtin proved that this critical value is $1/2$ [6]. Later, P.Erdős and J.Spencer proved that for $p = 1/2$ the probability that $Q_{p_n}^n$ is connected, as $n \rightarrow \infty$, tends to e^{-1} [4].

In the first section of this chapter, first the probability that $Q_{p_n}^n$ has no isolated point as $n \rightarrow \infty$, for fixed p , is investigated. It is proved that for $p < 1/2$ the probability that $Q_{p_n}^n$ has no isolated point, as $n \rightarrow \infty$, tends to 0. Therefore, for $p < 1/2$ the probability that $Q_{p_n}^n$ is connected, as $n \rightarrow \infty$, tends to 0. Then it is proved that, for $p > 1/2$ the probability that $Q_{p_n}^n$ has no isolated point, as $n \rightarrow \infty$, tends to 1. In the next step, the probability that $Q_{p_n}^n$ has no isolated point, as $n \rightarrow \infty$, when p depends on n and it is close $1/2$, is explored. It is proved that for $\lambda(n) = \lambda > 0$ and $p_n = 1 - (1/2)\lambda^{1/n}(1 + o(1/n))$, the probability that $Q_{p_n}^n$ has no isolated point, as $n \rightarrow \infty$, tends to $e^{-\lambda}$ [1]. Finally, it is proved that, for fixed $p = 1/2$ the probability that $Q_{p_n}^n$ has no isolated point, as $n \rightarrow \infty$, tends to e^{-1} . These results are based on P.Erdős and J.Spencer's work [4].

In the second section, one sheds light on the Isoperimetric problem, which is the problem of finding an inequality which relates the size of a subgraph to the size of its boundary. The solution to this problem for $Q_{p_n}^n$ is presented by S.Hart [5]. One needs such an inequality to explore the probability that $Q_{p_n}^n$ has a component which is not the whole graph.

In the last section, the Isoperimetric inequality is applied to prove that when p depends on n and $p_n \geq 1 - (1/2)(\log n)^{1/n}$, then the probability that there are no components with size larger than 1 and smaller than 2^n in $Q_{p_n}^n$, as $n \rightarrow \infty$, tends to 1. Therefore, for $p_n = 1 - (1/2)\lambda^{1/n}(1 + o(1/n))$, the probability that $Q_{p_n}^n$ is connected, as $n \rightarrow \infty$, tends to $e^{-\lambda}$. Finally, as a special case, it is shown that, for fixed p if $p = 1/2$, the probability that $Q_{p_n}^n$ is

¹The proof presented in this chapter is based on the proof presented by B.Bollobás in [1] p.384-393.

connected, as $n \rightarrow \infty$, tend to e^{-1} ; and if $p > 1/2$, the probability that $Q_{p_n}^n$ is connected, as $n \rightarrow \infty$, tends to 1.

4.1 Isolated vertices

For $p < 0.5$:

Assume p is fixed and $p < 0.5$. First, consider the following definitions:

Definition 14. $f_n(p_n) := \mathbf{P}(Q_{p_n}^n \text{ is connected})$

Definition 15. $g_n(p_n) := \mathbf{P}(Q_{p_n}^n \text{ contains an isolated point})$

Definition 16. $X_i(n) := \begin{cases} 1 & \text{Vertex } i \text{ is isolated, } i \in V(Q_{p_n}^n); \\ 0 & \text{Vertex } i \text{ is NOT isolated, } i \in V(Q_{p_n}^n). \end{cases}$, and $X(n) := \sum_{i \in V(Q_{p_n}^n)} X_i(n)$.

Now, calculate $E[X(n)]$ and $Var[X(n)]$ as follows:

$$\mu := E[X(n)] = \sum_{i \in V(Q_{p_n}^n)} E[X_i(n)] = \sum_{i \in V(Q_{p_n}^n)} (1-p)^n = 2^n(1-p)^n \quad (4.1)$$

$$Var[X(n)] = \sum_{i \in V(Q_{p_n}^n)} Var[X_i(n)] + \sum_{i \neq j; i, j \in V(Q_{p_n}^n)} Cov[X_i(n), X_j(n)] \quad (4.2)$$

where, $Var[X_i(n)]$ and $Cov[X_i(n), X_j(n)]$ are equal to:

$$\sum_{i \in V(Q_{p_n}^n)} Var[X_i(n)] = 2^n(1-p)^n - 2^n(1-p)^n(1-p)^n = \mu - \mu(1-p)^n \quad (4.3)$$

$$Cov[X_i(n), X_j(n)] = E[X_i(n)X_j(n)] - E[X_i(n)]E[X_j(n)] \quad (4.4)$$

$$= \begin{cases} 0 & \text{i, j not adjacent;} \\ (1-p)^n(1-p)^{n-1} - (1-p)^n(1-p)^n = \frac{\mu^2}{2^{2n}}(\frac{p}{1-p}) & \text{i, j adjacent.} \end{cases} \quad (4.5)$$

and finally:

$$Var[X(n)] = \mu - \mu(1-p)^n + \frac{\mu^2}{2^n}(\frac{np}{1-p}) = \mu + \mu(1-p)^n(\frac{np}{1-p} - 1) \quad (4.6)$$

Now, since we have $Var[X(n)]$, we can use Chebyshev's inequality to estimate $g_n(p)$. Chebyshev's inequality states that:

$$1 - g_n(p) = \mathbf{P}[X(n) = 0] \leq \mathbf{P}[|X(n) - \mu| \geq \mu] \leq \frac{Var[X(n)]}{\mu^2} \quad (4.7)$$

By applying Chebyshev's inequality when $p < 0.5$, one gets $Var[X(n)]/\mu^2 \rightarrow 0$, as $n \rightarrow \infty$. Therefore $\lim_{n \rightarrow \infty} g_n(p) = 1$. And finally, since $f_n(p) \leq 1 - g_n(p)$, then for $p < 0.5$ the probability that $Q_{p_n}^n$ is connected for $p < 0.5$, as $n \rightarrow \infty$, tends to 0. ■

For $p > 0.5$:

Assume p is fixed and $p > 0.5$. In order to calculate $g_n(p)$ when $p > 0.5$, as $n \rightarrow \infty$, one can use the following inequality:

$$g_n(p) = \mathbf{P}[X(n) > 0] \leq E[X(n)] = \mu \quad (4.8)$$

Since $E[X(n)] \rightarrow 0$ as $n \rightarrow \infty$, then $\lim_{n \rightarrow \infty} g_n(p) = 0$. This means that the probability that there are no isolated points in $Q_{p_n}^n$ for $p > 0.5$, as $n \rightarrow \infty$, tends to 1. ■

For $p_n = 1 - (1/2)\lambda^{1/n}(1 + o(1/n))$:

One needs the following theorem from [1] to find the distribution of $X(n)$ (distribution of the number of isolated points).

Theorem 7. Let $\lambda = \lambda(n)$ be a non-negative bounded function on \mathbf{N} . Suppose the non-negative integer valued random variables $X(1), X(2), \dots$ are such that:

$$\lim_{n \rightarrow \infty} \{E_r[X(n)] - \lambda^r\} = 0, \quad r = 0, 1, \dots \quad (4.9)$$

where $E_r[X]$ is the r th factorial moment of X , i.e. $E_r[X] = E[(X)_r]$. Then

$$X(n) \xrightarrow{d} \mathbf{P}_\lambda \quad (4.10)$$

Use the definition of $X(n)$ presented in definition 16. The goal is to calculate $E[X(n)]$.

$$E_r[X(n)] = E[X(n)(X(n) - 1)(X(n) - 2) \dots (X(n) - r + 1)] \quad (4.11)$$

Since $X(n) := \sum_{i \in V(Q_{p_n}^n)} X_i(n)$ and X_i 's are indicator functions, therefore:

$$X(n)(X(n) - 1)(X(n) - 2) \dots (X(n) - r + 1) = \sum_{(i_1, i_2, \dots, i_r)} X_{i_1} X_{i_2} \dots X_{i_r} \quad (4.12)$$

where the sum is over all ordered sets of distinct vertices. Then:

$$E_r[X(n)] = E[X(n)(X(n) - 1)(X(n) - 2) \dots (X(n) - r + 1)] \quad (4.13)$$

$$= E\left[\sum_{(i_1, i_2, \dots, i_r)} X_{i_1} X_{i_2} \dots X_{i_r}\right] \quad (4.14)$$

$$= \sum_{(i_1, i_2, \dots, i_r)} \mathbf{P}[X_{i_1} = 1, X_{i_2} = 1, \dots, X_{i_r} = 1] \quad (4.15)$$

One knows that a set of r vertices is incident with at most rn edges. There are $(r)_r \binom{2^n}{r}$ ways to choose such r vertices. Hence:

$$E_r[X(n)] \geq (r)_r \binom{2^n}{r} (1 - p_n)^{rn} = (2^n)_r (1 - p_n)^{rn} \quad (4.16)$$

On the other hand, a set of r vertices is incident with at least $r(n - r)$ edges. There are at most $(r - 1)_{r-1} \binom{2^n}{r-1} (r - 1)n$ ways to choose a set of r vertices in $Q_{p_n}^n$ where at least two vertices are adjacent; since if we choose $r - 1$ vertices independently, then the last vertex must be connected to one of the chosen vertices. In other words, there are at most $(r - 1)_{r-1} \binom{2^n}{r-1} (r - 1)n$ ways to choose r vertices which some of them are adjacent to each other. Hence:

$$E_r[X(n)] \leq (2^n)_r (1 - p_n)^{rn} + (r - 1)_{r-1} \binom{2^n}{r-1} (r - 1)n (1 - p_n)^{r(n-r)} \quad (4.17)$$

$$\leq (2^n)_r (1 - p_n)^{rn} + (2^n)_r rn (1 - p_n)^{r(n-r)} \quad (4.18)$$

$$\leq (2^n)_r (1 - p_n)^{rn} + 2^{n(r-1)} rn (1 - p_n)^{r(n-r)} \quad (4.19)$$

Finally from 4.16 and 4.19 one gets:

$$(2^n)_r(1-p_n)^{rn} \leq E_r[X(n)] \leq (2^n)_r(1-p_n)^{rn} + 2^{n(r-1)}rn(1-p_n)^{r(n-r)} \quad (4.20)$$

$$(4.21)$$

which gives:

$$(2(1-p_n))^{rn}(1-\frac{r}{2^n})^r \leq E_r[X(n)] \leq (2(1-p_n))^{rn}\{1+2^{-n}rn(1-p_n)^{-r^2}\} \quad (4.22)$$

Since r is fixed and $\lim_{n \rightarrow \infty} (2(1-p_n))^n = \lambda$, then:

$$\lim_{n \rightarrow \infty} (E_r[X(n)]) = \lambda^r \text{ for } r=0,1,2,\dots \quad (4.23)$$

This shows that $X(n) \xrightarrow{d} \mathbf{P}_\lambda$. ■

For $p = 0.5$:

In the calculation of $p_n = 1 - 1/2\lambda^{1/n}(1 + o(1/n))$, if we fix $p = 1/2$ and let $\lambda = 1$, then we get that the distribution of $X(n)$, as $n \rightarrow \infty$, tends to a Poisson distribution with mean 1. Therefore, one can conclude:

$$\lim_{n \rightarrow \infty} (1 - g_n(p)) = \lim_{n \rightarrow \infty} (\mathbf{P}(X(n) = 0)) = e^{-1} \quad (4.24)$$

This shows that for $p = 1/2$ the probability that $Q_{p_n}^n$ has no isolated point, as $n \rightarrow \infty$, tends to e^{-1} . ■

4.2 Isoperimetric problem for the cube

One needs an inequality which relates the size of a subgraph of Q^n to the size of its boundary. This inequality will be applied to prove that for fixed values of p if $p \geq 0.5$, then the probability that subgraphs of Q^n do not have a component of size larger than 2 and smaller than 2^n , as $n \rightarrow \infty$, tends to 1. The proof presented here is based on the proof presented in [1].

Definition 17. The edge boundary $b_G(H)$, where H is an induced subgraph of G , is the number of edges which joins vertices in H to the vertices in $G \setminus H$.

Definition 18. $b_G(m) := \min\{b_G(H), H \text{ is an induced subgraph of } G, |V(H)| = m\}$.

The main task in this section is to calculate $b_{Q^n}(m)$. The answer, loosely, is if $m = 2^k$ for some $k < n$ then one should take a k -dimensional sub-cube of Q^n as $b_{Q^n}(H)$. If $2^k \leq m < 2^{k+1}$, for some $k < n$, then one should choose one side of a $(k+1)$ -cube and $m - 2^k$ more vertices properly chosen in the other half. Since Q^n is n -regular and H is an induced subgraph of G with $|V(H)| = m$, then:

$$b_{Q^n}(H) = mn - 2e(H), \text{ where } e(H) \text{ is the total number of edges in } H. \quad (4.25)$$

$$b_{Q^n}(m) = mn - 2e_n(m), \text{ where } e_n(m) = \max\{e(H) : H \text{ induced subgraph of } Q^n, |V(H)| = m\}. \quad (4.26)$$

Definition 19. $h(i) :=$ sum of digits in the binary expansion of i and $f(l, m) := \sum_{l \leq i < m} h(i)$

Lemma 1. If $1 \leq k \leq l$, then $f(l, l+k) \geq f(0, k) + k$

PROOF:

Let look at the binary expansion of a few numbers:

Column	3	2	1	0
Bin \ Dec	2^3	2^2	2^1	2^0
0				0
1				1
2			1	0
3			1	1
4		1	0	0
5		1	0	1
6		1	1	0
7		1	1	1
8	1	0	0	0
9	1	0	0	1
10	1	0	1	0
11	1	0	1	1

From this representation, one can observe that column i starts with a block of 2^i zeros. Therefore, sum of j th digits of k consecutive numbers is minimal if the first block of 0's is as long as possible. Hence, one can conclude:

$$f(l, l+k) \geq f(0, k) \quad (4.27)$$

For every i define r such that $0 \leq i \leq 2^r - 1$. The binary expansions of i and $2^r - 1 - i$ are symmetric. This means that, if there is a 1/0 in an specific location of the binary expansion of i then there is a 0/1 in the same location of the binary expansion of $2^r - 1 - i$. Therefore,

$$h(i) + h(2^r - 1 - i) = r \text{ for } 0 \leq i \leq 2^r - 1 \quad (4.28)$$

Consequently, since:

$$\sum_{l \leq i < l+k} h(i) + \sum_{2^r - l - k \leq i < 2^r - l} h(i) = rk \quad (4.29)$$

then:

$$f(l, l+k) + f(2^r - l - k, 2^r - l) = rk, \text{ if } l+k \leq 2^r \quad (4.30)$$

Let us prove lemma 1 with the assumption $k \leq 2^r \leq l$ by using inequalities 4.27 and 4.30. This assumption means that the length of the sequence in the binary expansion of $2^r + k$ and 2^r are equal.

With the same logic that one gets 4.27, one gets:

$$f(l, l+k) \geq f(2^r, 2^r + k) \text{ when } 2^r \leq l \quad (4.31)$$

and then for $k \leq 2^r$ one can get:

$$f(2^r, 2^r + k) = \sum_{2^r \leq i < 2^r + k} h(i) \quad (4.32)$$

$\sum_{2^r \leq i < 2^{r+k}} h(i)$ is the sum over numbers with the same length in their binary expansion's sequence. When one removes the last digit in their binary expansion, the remain is $f(0, k)$. Therefore, $\sum_{2^r \leq i < 2^{r+k}} h(i)$ is equal to k 1's plus $f(0, k)$. Hence:

$$f(2^r, 2^r + k) = k + f(0, k) \text{ when } k \leq 2^r \leq l \quad (4.33)$$

and finally:

$$f(l, l + k) \geq f(0, k) + k \text{ where } k \leq 2^r \leq l \quad (4.34)$$

Now, one can prove lemma 1 by induction on K , without the assumption $k \leq 2^r \leq l$. We want to prove that for $1 \leq K \leq l$, $f(l + K, l) \geq K + f(0, K)$. Fix k such that $1 \leq k \leq l$ and $K < k$. For $K = 1$ the inequality in lemma 1 is trivial. Assume that the inequality is true for $K < k$ and $K > 2$, which means:

$$f(l, l + K) \geq K + f(0, K) \text{ when } 1 \leq k \leq l, \text{ and } K < k \quad (4.35)$$

Now, one should verify the inequality for $K = k$. Define $r \geq 1$ by $2^{r-1} \leq k < 2^r$. If $l \geq 2^r$, then $k \leq 2^r \leq l$ and the lemma is implied by inequality 4.34. Hence, one may assume that $2^{r-1} < l < 2^r$. Now, one should apply inequality 4.30 and 4.35 in order to get the final result:

$$f(l + k) = f(l, 2^r) + f(2^r, l + k) \text{ (from definition of } f \text{ and } l \geq 2^r) \quad (4.36)$$

$$= (2^r - l)r - f(0, 2^r - l) + f(2^r, l + k) \text{ (from 4.30)} \quad (4.37)$$

$$\geq (2^r - l)r - f(0, 2^r - l) + f(0, l + k - 2^r) + l + k - 2^r \text{ (from 4.35)} \quad (4.38)$$

$$\geq (2^r - l)r - f(2^r - k, 2^r - k + 2^r - l) + 2^r - l + f(0, l + k - 2^r) + l + k - 2^r \text{ (from 4.35)} \quad (4.39)$$

$$\geq (2^r - l)r - f(2^r - k, 2^r - k + 2^r - l) + f(0, l + k - 2^r) + k \quad (4.40)$$

$$\geq f(l + k - 2^r, k) + f(0, l + k - 2^r) + k \text{ (from 4.30)} \quad (4.41)$$

$$\geq f(0, k) + k \text{ (from characteristics of } f) \quad (4.42)$$

■

Theorem 8. For $2 \leq m \leq 2^n$ we have $b_{Q^n}(m) = mn - 2f(0, m)$. In other words, $f(0, m) = e_n(m)$ where $e_n(m) = \max\{e(H) : H \text{ induced subgraph of } Q^n, |V(H)| = m\}$.

PROOF:

First, let us fix an m . As the first step one should prove that $e_n(m) \geq f(0, m)$. Vertex i is connected to $h(i)$ vertices j with $j < i$, since for each 1 in the binary expansion of i there is exactly one j ($j < i$), which its binary expansion differs in the position of that 1. Therefore, one can conclude that $W = \{0, 1, 2, \dots, m - 1\}$ contains $\sum_{0 \leq i < m} h(i) = f(0, m)$ edges. So, $e_n(m) \geq f(0, m)$.

As the second step, one should prove that $e_n(m) \leq f(0, m)$ by induction on n . Fix m and n for $2 \leq m \leq 2^n$. For $n = 1$ the inequality is trivially true. Assume that it is true for $N < n$, which means:

$$e_N(m) \leq f(0, m), \text{ where } N < n \text{ and the fixed } m \text{ is: } 2 \leq m \leq 2^n \quad (4.43)$$

Now, one should check the inequality 4.43 for $N = n$. This means that we should find an H induced subgraph of Q^n , $|V(H)| = m$, which maximize $e_n(m)$. Let us split Q^n into two $(n-1)$ -dimensional cubes, the top face with 2^{n-1} vertices and the bottom face with 2^{n-1} vertices. This means, there are $(n-1)2^{n-2}$ edges in each face, and 2^{n-1} edges between two faces. Now, one can construct H . Choose m_1 vertices for H from the top face, and m_2 vertices from the

bottom face, where $m_1 + m_2 = m$ and $m_1 \leq m_2$. In other words, H is constructed from two induced subgraphs, one from the top face, denoted by H_1 , and the other from the bottom face, denoted by H_2 .

Each face is a $(n-1)$ -dimensional cube, so inequality 4.43 holds for both H_1 and H_2 . Also, each vertex of the top face is connected to exactly one vertex of the bottom face. Hence, the number of edges of H is at most:

$$e_n(m) \leq f(0, m_1) + f(0, m_2) + m_1 \text{ (from 4.43)} \quad (4.44)$$

where m_1 , in the right hand side of the inequality, is for the maximum number of edges between H_1 and H_2 , which one can choose here. Finally, by applying lemma 1, one gets:

$$e_n(m) \leq f(0, m_1) + f(0, m_2) + m_1 \quad (4.45)$$

$$\leq f(m_2, m_2 + m_1) + f(0, m_2) \text{ (from lemma 1)} \quad (4.46)$$

$$\leq f(0, m) \text{ (from definition of f)} \quad (4.47)$$

■

Theorem 8 shows that, if we want to choose an induced subgraph of Q^n , with m vertices, which has the smallest edge boundary, then we should choose the induced subgraph of Q^n with the set of vertices $W = \{0, 1, 2, \dots, m-1\}$.

Corollary 1. For all k and n , $e_n(k) \leq \frac{k}{2} \lceil \log_2 k \rceil$, which is equivalent to $b_{Q^n}(k) \geq k(n - \lceil \log_2 k \rceil)$.

PROOF:

Let $r = \lceil \log_2 k \rceil$. Then

$$2f(0, k) \leq f(0, k) + f(0, k) \leq f(0, k) + f(2^r - k, 2^r) \text{ (from 4.27)} \quad (4.48)$$

$$= rk \text{ (from 4.30)} \quad (4.49)$$

Therefore:

$$e_n(k) = f(0, k) \leq r \frac{k}{2} = \frac{k}{2} \lceil \log_2 k \rceil \quad (4.50)$$

■

4.3 Isolated components of size larger than 2 and smaller than 2^n

Definition 20. C_s is the family of s -subsets (subsets with size s) of $V = V(Q^n)$ whose induced graph is connected.

Remarks: $h(n) := o(g(n))$ means $\frac{h(n)}{g(n)} \rightarrow 0$ as $n \rightarrow \infty$.

Remarks: The following inequality will be applied a lot in the rest of this section:

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \frac{n^k}{k!} \leq \left(\frac{ne}{k}\right)^k \quad (4.51)$$

Theorem 9. If $p_n \geq 1 - \frac{1}{2}(\log n)^{\frac{1}{n}}$, the probability that for some $S \in C_s$, $2 \leq s \leq 2^{n-1}$, no edges of $Q_{p_n}^n$ join S to $V(Q^n) \setminus S$, as $n \rightarrow \infty$, tends to 0.

Note: For $2^{n-1} < s < 2^n$, if there exist a component of size smaller than 2^n then there is at least one component of size smaller than 2^{n-1} which contradicts with the theorem.

PROOF:

Consider $S \subset V = V(Q^n)$ and set $b(S) = b_{Q^n}(H)$ for which H is the induced subgraph of Q^n with the set of vertices S . One can observe that:

$$\mathbf{P}(\text{No edges of } Q_{p_n}^n \text{ join } S \text{ to } V \setminus S) = (1 - p_n)^{b(S)} \quad (4.52)$$

In order to prove the theorem, it is sufficient to show that:

$$\sum_{s=2}^{2^{n-1}} \sum_{S \in C_s} (1 - p_n)^{b(S)} = o(1) \quad (4.53)$$

From corollary 1, one knows that for $|S| = s$:

$$b(S) \geq b(s) \geq s(n - \lceil \log_2 s \rceil) \quad (4.54)$$

and therefore:

$$\sum_{S \in C_s} (1 - p_n)^{b(S)} \leq |C_s| (1 - p_n)^{b(s)} \quad (4.55)$$

Now, one should partition s , $2 \leq s \leq 2^{n-1}$, to different intervals in order to find a bound for $|C_s|$ and $(1 - p_n)^{b(s)}$ for each interval.

First interval $2 \leq s \leq s_1$, $s_1 = \lfloor \frac{2^{\frac{n}{2}}}{n^2} \rfloor$:

First, one should find a bound for $|C_s|$. One has maximum 2^n choices to choose the first element for C_s . The selected element is connected to maximum n vertices, therefore there are n choices to choose the second element. With the same logic there are at most $(s-1)n$ choices to choose the last element for C_s . Therefore, one can show:

$$|C_s| \leq 2^n(n)(2n)\dots((s-1)n) \leq (s-1)!(n)^{s-1}2^n \quad (4.56)$$

Hence:

$$|C_s| (1 - p_n)^{b(s)} \leq (s-1)!(n)^{s-1}2^n (1 - p_n)^{s(n - \lceil \log_2 s \rceil)} \quad (4.57)$$

Since $p_n = 1 - \frac{1}{2}(\log n)^{\frac{1}{n}}$, so for large enough n :

$$(1 - p_n)^{s(n - \lceil \log_2 s \rceil)} \leq (2)^{-ns} (\log n)^s (1 - p_n)^{-s(\log_2 s)} (\text{neglecting some small terms}) \quad (4.58)$$

$$= (2)^{-ns} (\log n)^s 2^{s \log_2 s} (\log n)^{\frac{-s \log_2 s}{n}} \quad (4.59)$$

$$(\text{ since for large enough } n: (\log n)^{\frac{-s \log_2 s}{n}} \leq 1) \quad (4.60)$$

$$\leq (2)^{-ns} (\log n)^s s^s \quad (4.61)$$

From equations 4.57 and 4.61, one can show that:

$$|C_s| (1 - p_n)^{b(s)} \leq (s-1)!(n)^{s-1}2^n (2)^{-ns} (\log n)^s s^s \quad (4.62)$$

Assume that the right hand side of inequality 4.62 is equal to A . After multiplying both sides of inequality 4.62 with $\frac{n s^{s+1}}{s!}$ and then taking \log_2 from both sides, one gets:

$$\log_2(|C_s| (1 - p_n)^{b(s)} \frac{n s^{s+1}}{s!}) \leq \log_2(A \frac{n s^{s+1}}{s!}) \quad (4.63)$$

If $\log_2(A \frac{n s^{s+1}}{s!}) \rightarrow -\infty$ as $n \rightarrow \infty$ then $A \frac{n s^{s+1}}{s!}$ should tend to 0. This means that $|C_s|(1 - p_n)^{b(s)} \frac{n s^{s+1}}{s!}$ tends to 0, as $n \rightarrow \infty$. Therefore:

$$|C_s|(1 - p_n)^{b(s)} \leq \frac{s!}{n s^{s+1}} \text{ for large enough } n \quad (4.64)$$

which shows that:

$$\sum_{s=2}^{s_1} \sum_{S \in C_s} (1 - p_n)^{b(S)} = o(1) \quad (4.65)$$

Finally, it remains to prove $\log_2(A \frac{n s^{s+1}}{s!}) \rightarrow -\infty$ as $n \rightarrow \infty$. One can verify this for $s \leq n$ and $s > n$.

Second interval $s_1 + 1 \leq s \leq 2^{n-1}$ and $S \in C_s^-, s_1 = \lfloor \frac{2^n}{n^2} \rfloor$:

Let us define C_s^- and C_s^+ as follows:

Definition 21.

$$C_s^- := \{S \in C_s | b(s) \geq s(n - \log_2 s + \log_2 n)\}, \text{ and } C_s^+ := C_s \setminus C_s^- \quad (4.66)$$

One can bound $|C_s^-|$ for $s_1 + 1 \leq s \leq 2^{n-1}$ as follows:

$$|C_s^-| \leq |C_s| \leq \binom{2^n}{s} \leq \frac{2^{ns}}{s!} \leq \left(\frac{e 2^n}{s}\right)^s \quad (4.67)$$

Hence:

$$\sum_{s=s_1+1}^{2^{n-1}} \sum_{S \in C_s^-} (1 - p_n)^{b(S)} \leq \sum_{s=s_1+1}^{2^{n-1}} \left(\frac{e 2^n}{s}\right)^s \left(\frac{1}{2}(\log n)^{\frac{1}{n}}\right)^{s(n - \log_2 s + \log_2 n)} \quad (4.68)$$

$$\leq \sum_{s=s_1+1}^{2^{n-1}} \left(\frac{e 2^n 2^{-(n - \log_2 s + \log_2 n)} (\log n)^{\frac{(n - \log_2 s + \log_2 n)}{n}}}{s}\right)^s \quad (4.69)$$

$$\leq \sum_{s=s_1+1}^{2^{n-1}} \left(\frac{e 2^n 2^{-n \log_2 s} 2^{-\log_2 n} \log n}{s}\right)^s (\log n)^{\frac{n(-\log_2 s + \log_2 n)}{s}} \quad (4.70)$$

$$(\text{ since for large enough } n: (\log n)^{\frac{n(-\log_2 s + \log_2 n)}{s}} \leq 1) \quad (4.71)$$

$$\leq \sum_{s=s_1+1}^{2^{n-1}} \left(\frac{e \log n}{n}\right)^s = o(1) \quad (4.72)$$

■

Third interval $s_1 \leq s \leq s_2$, $s_1 = \lfloor \frac{2^n}{n^2} \rfloor$, $s_2 = \lfloor \frac{2^n}{(\log n)^4} \rfloor$ and $S \in C_s^+$:

For the 3rd and the 4th intervals one needs to know how to find a bound for $|C_s^+|$. The following lemma, presented by B. Bollobas [1], helps us in this matter:

Lemma 2. Let G be a graph of order v and suppose that $\Delta(G) \leq \Delta$, $2e(G) = vd$ and $\Delta + 1 \leq u \leq v - \Delta - 1$. Then, there is a u -set of U of vertices with:

$$|N(U)| = |U \cup \Gamma(U)| \geq v \frac{d}{\Delta} \left\{1 - \exp\left(\frac{-u(\Delta + 1)}{v}\right)\right\} \quad (4.73)$$

where, $\Delta(G) :=$ Maximum degree in G , $d :=$ average degree in G and $\Gamma(U) = \{x \in V(G) : xy \in E(G) \text{ for some } y \in U\}$

Let $H = Q_n[S]$ (the induced subgraph of Q_n with the set of vertices S). From the definition of C_s^+ one knows that the average degree in H is at least:

$$\log_2 s - \log_2 n \quad (4.74)$$

The goal is to find $U \subset S$, where $|U| := u := \lfloor \frac{2s}{n} \rfloor$, $\Delta = n$, $v = s$, $d \geq \log_2 s - \log_2 n$ and then use lemma 2 to calculate $|N(U)|$. First, one should check the condition $\Delta + 1 \leq u \leq v - \Delta - 1$ for defined variables in order to use lemma 2. First, check if $n + 1 \leq \lfloor \frac{2s}{n} \rfloor$, as $n \rightarrow \infty$:

$$\frac{2s}{n} = \frac{2^{\frac{n}{2}+1}}{n^3} \text{ for minimum } s, \text{ and trivially } n + 1 \leq \frac{2^{\frac{n}{2}+1}}{n^3} \text{ for large enough } n \quad (4.75)$$

$$(4.76)$$

and then check if $\lfloor \frac{2s}{n} \rfloor \leq s - (n + 1)$. One should check if $ns - n(n + 1) \geq 2s$, which means one should check that whether:

$$\frac{2^{\frac{n}{2}}(n - 2)}{n^3(n + 1)} \geq 1 \quad (4.77)$$

which is clearly true for large enough n . Now, one can apply lemma 2 on the graphs generated by S and get :

$$\exists U \subset S : |N(U)| \geq s \frac{\log_2 s - \log_2 n}{n} \{1 - \exp(-\frac{u(n + 1)}{s})\} \quad (4.78)$$

where:

$$\frac{\log_2 s - \log_2 n}{n} \geq \frac{(\log_2(\frac{2^{\frac{n}{2}}}{n^2}) - \log_2 n)}{n} = \frac{n - 6 \log_2 n}{2n} \quad (4.79)$$

$$\text{and } \lim_{n \rightarrow \infty} \frac{n - 6 \log_2 n}{2n} = \frac{1}{2} \quad (4.80)$$

on the other hand:

$$\lim_{n \rightarrow \infty} (1 - \exp(-\frac{n + 1}{s}(\frac{2s}{n} + 1))) = 1 - e^{-2} \quad (4.81)$$

Therefore, from 4.80 and 4.81 one gets:

$$|N(U)| \geq \frac{1}{2}(1 - e^{-2})s \geq \frac{s}{3} \text{ as } n \rightarrow \infty \quad (4.82)$$

Now that we have $|N(U)|$, we can estimate a bound for $|C_s^+|$ here. We know from 4.82 that for each $S \in C_s^+$ there exist a $U \subseteq S$, $|U| := u := \lfloor \frac{2s}{n} \rfloor$, such that $|N(U)| \geq s/3$. Therefore, one can choose $S \in C_s^+$ as follows:

1. Select u vertices of Q^n ; there are $\binom{2^n}{u}$ choices for this u .
2. Select $\lfloor \frac{s}{3} \rfloor - u$ neighbors of the selected vertices in part 1; there are maximum $(2^n)^u$ choices, since there are at most $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$ ways to find neighbors of a vertex in U .
3. Select $\lfloor \frac{2s}{3} \rfloor$ other vertices; there are at most $\binom{2^n}{\lfloor \frac{2s}{3} \rfloor}$ choices.

Hence:

$$|C_s^+| \leq \binom{2^n}{u} (2^n)^u \binom{2^n}{\lfloor \frac{2s}{3} \rfloor} \quad (4.83)$$

and:

$$\sum_{S \in C_s^+} (1 - p_n)^{b(S)} \leq \binom{2^n}{u} (2^n)^u \binom{2^n}{\lfloor \frac{2s}{3} \rfloor} (1 - p_n)^{b(s)} \quad (4.84)$$

where:

$$(1 - p_n)^{b(s)} \leq 2^{-sn} s^s (\log n)^s \quad (4.85)$$

consequently from 4.84, 4.85 and 4.51:

$$\sum_{S \in C_s^+} (1 - p_n)^{b(S)} \leq \left(\frac{e2^n}{u}\right)^u 2^{un} \left(\frac{e2^n}{\lfloor \frac{2s}{3} \rfloor}\right)^{\lfloor \frac{2s}{3} \rfloor} 2^{-sn} s^s (\log n)^s \quad (4.86)$$

Write $s = 2^{\beta n}$, ($\beta = \frac{\log_2 s}{n}$), so that:

$$2^{\beta n} \leq \frac{2^n}{(\log n)^4} \Rightarrow \beta \leq 1 - \frac{4 \log_2 \log n}{n} \quad (4.87)$$

Now, find a bound for the inequality 4.86. First calculate the first part of the inequality:

$$\left(\frac{e2^n}{u}\right)^u 2^{un} \left(\frac{e2^n}{\lfloor \frac{2s}{3} \rfloor}\right)^{\lfloor \frac{2s}{3} \rfloor} \leq \left(\frac{e2^n}{\frac{2s}{n}}\right)^{\frac{2s}{n}} 2^{2s} \left(\frac{e2^n}{\frac{2s}{3}}\right)^{\frac{2s}{3}} = (2^2 2^2 \left(\frac{3}{2} e\right)^{\frac{2}{3}})^s \frac{2^{\frac{2s}{3n}}}{s^{\frac{2s}{3}}} \left(\frac{2^n}{s}\right)^{\frac{2s}{3}} \quad (4.88)$$

$$\left(\text{since for large enough } n \text{ and } s_1 \leq s \leq s_2: \left(\frac{e2^n}{\frac{2s}{3}}\right)^{\frac{2s}{3}} \leq 1\right) \quad (4.89)$$

$$= (2^2 2^2 \left(\frac{3}{2} e\right)^{\frac{2}{3}})^s \frac{2^{\frac{2s}{3n}}}{s^{\frac{2s}{3}}} = c^s 2^{\frac{2}{3} sn (1 - \frac{\log_2 s}{n})} = c^s 2^{\frac{2}{3} sn (1 - \beta)} \quad (4.90)$$

where c is a positive constant. Now, by substituting 4.90 in 4.86 one gets:

$$\sum_{S \in C_s^+} (1 - p_n)^{b(S)} \leq 2^{-sn} s^s (\log n)^s c^s 2^{\frac{2}{3} sn (1 - \beta)} \quad (4.91)$$

$$= c^s (\log n)^s 2^{-\frac{sn(1-\beta)}{3}} \quad (4.92)$$

$$\leq c^s (\log n)^s 2^{-\frac{4s \log_2 \log n}{3n}}, \text{ (from 4.87)} \quad (4.93)$$

$$= c^s (\log n)^s 2^{\log_2 (\log n) \frac{-4s}{3}} \quad (4.94)$$

$$\leq c^s (\log n)^s (\log n)^{\frac{-4s}{3}} \quad (4.95)$$

$$= c^s (\log n)^{\frac{-s}{3}} \quad (4.96)$$

and finally from 4.96:

$$\sum_{s=s_1}^{s_2} \sum_{S \in C_s^+} (1 - p_n)^{b(S)} \leq \sum_{s=s_1}^{s_2} c^s (\log n)^{\frac{-s}{3}} = o(1) \quad (4.97)$$

■

Fourth interval $s_2 + 1 \leq s \leq 2^{n-1}$ and $s_2 = \lfloor \frac{2^n}{(\log n)^4} \rfloor, S \in C_s^+$:

In $H = Q^n[S]$ (the induced subgraph of Q^n with the set of vertices S), the average degree is at least:

$$\log_2 s - \log_2 n > n - 2 \log_2 n \quad (4.98)$$

since:

$$s \geq \lceil \frac{2^n}{(\log n)^4} \rceil \Rightarrow \log_2(\frac{2^n}{(\log n)^4}) < \log_2 s \quad (4.99)$$

$$\Rightarrow \log_2 s - \log_2 n \geq \log_2(\frac{2^n}{(\log n)^4}) - \log_2 n \geq n - \log_2(\log n)^4 - \log_2 n \quad (4.100)$$

$$(\text{for large enough } n \text{ one can get, } n > (\log n)^4) \quad (4.101)$$

$$\geq n - 2 \log_2 n \quad (4.102)$$

First, look for a subgraph of H with large average degree. Let T be the set of vertices of H with degree at least $n - (\log_2 n)^2$, and set $t = |T|$. From 4.98 one can conclude that the sum of degrees in H is at least $s(n - 2 \log_2 n)$. We also know that:

$$\text{Sum of degrees in } S \leq s(n - 2 \log_2 n) \quad (4.103)$$

$$\leq t \times (\text{Maximum degree of vertices in set } T \text{ of graph } H) \quad (4.104)$$

$$+ (s - t) \times (\text{Maximum degree of vertices in set } S \setminus T \text{ of graph } H) \quad (4.105)$$

$$\leq tn + (s - t)(n - (\log_2 n)^2) \quad (4.106)$$

$$\Rightarrow t \geq s(1 - \frac{2}{\log_2 n}) \quad (4.107)$$

Define $H_1 = Q^n[T] = H[T]$ as the induced subgraph spanned by T . We want to calculate $|N_{H_1}(U)|$ for some U in H_1 , hence we should estimate the size of H_1 and after that calculate the average degree in T . Let us first calculate $e(H_1)$, the total number of edges in H_1 .

$$e(H_1) \geq e(H) - (s - t)n \geq \frac{s}{2}(n - 2 \log_2 n) - \frac{2s}{\log_2 n}n \quad (\text{from 4.98 and 4.107}) \quad (4.108)$$

One knows that the average degree in H_1 is at least $\frac{2e(H_1)}{s}$, and:

$$\frac{2e(H_1)}{s} \geq n - 2 \log_2 n - \frac{4n}{\log_2 n} \geq n - \frac{5}{\log_2 n} \quad (4.109)$$

$$(\text{since: } \log_2 n^2 < \frac{n}{\log_2 n} \text{ for large enough } n) \quad (4.110)$$

Set $u = \lfloor \frac{2^n}{n^{\frac{1}{2}}} \rfloor$. One should check the conditions of lemma 1 here. Let $v = t, \Delta = n, d \geq n - \frac{5}{\log_2 n}$. So, one should check if $n + 1 \leq \frac{2^n}{n^{\frac{1}{2}}} \leq t - (n + 1)$ for large enough n . Clearly, $n + 1 \leq \frac{2^n}{n^{\frac{1}{2}}}$, as $n \rightarrow \infty$. It remains to prove $\frac{2^n}{n^{\frac{1}{2}}} \leq t - (n + 1)$, for large enough n . For minimum s from 4.107 we can get:

$$t \geq \frac{2^n}{(\log n)^4} (1 - \frac{2}{\log_2 n}) \quad (\text{from 4.107}) \quad (4.111)$$

$$\geq \frac{2^n}{n^{\frac{1}{2}}} + n + 1 \quad (\text{for large enough } n) \quad (4.112)$$

Now, one can use lemma 1 and estimate $|N_{H_1}(U)|$.

$$|N_{H_1}(U)| \geq \frac{t}{n} (n - \frac{5n}{\log_2 n}) \{1 - \exp(-\frac{n+1}{t} \frac{2^n}{n^{\frac{1}{2}}})\} \quad (4.113)$$

$$\geq \frac{t}{2} (1 - \frac{5}{\log_2 n}) \{1 - \exp(-\frac{n+1}{t} \frac{2^n}{n^{\frac{1}{2}}})\} \quad (4.114)$$

After that, let us estimate a bound for $\exp(-\frac{n+1}{t} \frac{2^n}{n^{\frac{1}{2}}})$. One knows that $t \geq s(1 - \frac{2}{\log_2 n})$. Since $\max(t) = s$ and $\max(s) = 2^{n-1}$, then:

$$\frac{2^n(n+1)}{n^{\frac{1}{2}}t} \geq \frac{2^n(n+1)}{n^{\frac{1}{2}}2^{n-1}} = \frac{2(n+1)}{n^{\frac{1}{2}}} \geq n^{\frac{1}{4}} \text{ (for large enough } n) \quad (4.115)$$

$$\Rightarrow \{1 - \exp(-\frac{n+1}{t} \frac{2^n}{n^{\frac{1}{2}}})\} \geq \exp(-n^{\frac{1}{4}}) \text{ (for large enough } n) \quad (4.116)$$

By using the bound from 4.116 in 4.114, one gets:

$$|N_H(U)| \geq |N_{H_1}(U)| \geq t(1 - \frac{5}{\log_2 n})\{1 - \exp(-n^{\frac{1}{4}})\} \quad (4.117)$$

$$= t\{1 + \exp(-n^{\frac{1}{4}}) \frac{5}{\log_2 n} - \exp(-n^{\frac{1}{4}}) - \frac{5}{\log_2 n}\} \quad (4.118)$$

$$(\lim_{n \rightarrow \infty} \exp(-n^{\frac{1}{4}}) \frac{5}{\log_2 n} = 0 \text{ and } \exp(-n^{\frac{1}{4}}) < \frac{1}{\log_2 n} \text{ (for large enough } n)) \quad (4.119)$$

$$\geq t\{1 - \frac{1}{\log_2 n} - \frac{6}{\log_2 n}\} = t(1 - \frac{6}{\log_2 n}) \quad (4.120)$$

$$\geq s(1 - \frac{2}{\log_2 n})(1 - \frac{5}{\log_2 n}) = s(1 + \frac{2}{\log_2 n} \frac{6}{\log_2 n} - \frac{8}{\log_2 n}) \text{ (from 4.107)} \quad (4.121)$$

$$\geq s(1 - \frac{8}{\log_2 n}) \quad (4.122)$$

Now that we have $|N_H(U)|$, we can estimate a bound for $|C_s^+|$ here. We know from 4.122 that for each $S \in C_s^+$ there exist a $U \subseteq S$, $|U| := u := \lfloor \frac{2^n}{n^{\frac{1}{2}}} \rfloor$, such that $|N_H(U)| \geq s(1 - \frac{8}{\log_2 n})$.

Therefore, one can choose $S \in C_s^+$ as follows:

1. Select u vertices of Q^n ; there are $\binom{2^n}{u}$ choices for this u .
2. Select $\lfloor s(1 - \frac{8}{\log_2 n}) \rfloor - u$ neighbors of the selected vertices in part 1. At most $(\log_2 n)^2$ of the n neighbors of a vertex in U do not belong to $N_H(U)$. Hence there are at most $\sum_{(k_j)} (\prod_{i=1}^u \binom{n}{j})$ ways to find neighbors of u vertices in U , where the sum is over all (k_1, k_2, \dots, k_u) , $k_i \leq (\log_2 n)^2$. We know that:

$$\sum_{(k_i)} \prod_{i=1}^u \binom{n}{k_i} \leq \sum_{(k_i)} \prod_{i=1}^u \left(\frac{n^i}{i!} \right) \quad (4.123)$$

$$\leq \sum_{(k_i)} \prod_{i=1}^u \left(\frac{n^{(\log_2 n)^2}}{(\log_2 n)^{2!}} \right) \quad (4.124)$$

$$= \sum_{(k_i)} \frac{n^{u(\log_2 n)^2}}{((\log_2 n)^{2!})^u} \quad (4.125)$$

$$= ((\log_2 n)^2)^u \frac{n^{u(\log_2 n)^2}}{((\log_2 n)^{2!})^u} \quad (4.126)$$

$$\leq n^{u(\log_2 n)^2} \quad (4.127)$$

$$(4.128)$$

3. Select $\lfloor \frac{8s}{\log_2 n} \rfloor$ other vertices; there are at most $\binom{2^n}{\lfloor \frac{8s}{\log_2 n} \rfloor}$ choices.

Hence:

$$\sum_{S \in C_s^+} (1 - p_n)^{b(S)} \leq \binom{2^n}{u} n^{u(\log_2 n)^2} \left(\frac{2^n}{\lfloor \frac{8s}{\log_2 n} \rfloor} \right) 2^{-s(n - \log_2 s)} (\log n)^{s(1 - \frac{\log_2 s}{n})} \quad (4.129)$$

$$(4.130)$$

where:

$$\binom{2^n}{u} n^{u(\log_2 n)^2} \left(\frac{2^n}{\lfloor \frac{8s}{\log_2 n} \rfloor} \right) \leq 2^{o(s)} \quad (4.131)$$

Therefore:

$$\sum_{S \in C_s^+} (1 - p_n)^{b(S)} \leq 2^{\varepsilon(s)} \quad (4.132)$$

where:

$$\varepsilon(s) = o(s) - s\{n - \log_2 s - \log_2 \log n + \frac{\log_2 s}{n} \log_2 \log n\} \quad (4.133)$$

Since $s \leq 2^{n-1}$, hence one can get:

$$\varepsilon(s) \leq o(s) - s\{n - (n - 1) - \log_2 \log n + \frac{n - 1}{n} \log_2 \log n\} \quad (4.134)$$

$$= o(s) - s\{1 - \frac{1}{n} \log_2 \log n\} \leq \frac{-s}{2} \quad (4.135)$$

Therefore, for large enough n , one can get:

$$\sum_{s=s_2+1}^{2^{n-1}} \sum_{S \in C_s^+} (1 - p_n)^{b(S)} \leq \sum_{s=s_2+1}^{2^{n-1}} 2^{-\frac{s}{2}} = o(1) \quad (4.136)$$

■

4.4 Comparison between a typical random graph and Q^n

Consider $Per(G(V, E), p_n)$ as the subgraph of $G(V, E)$ after percolation with the parameter p_n . The goal in this section is to compare $\mathbf{P}(Q_{p_n}^n \text{ is connected})$ with $\mathbf{P}(Per(G \in G(N = 2^n, M = n2^{n-1}), p_n) \text{ is connected})$ as $n \rightarrow \infty$.

One knows, the probability that $G_p \in G(n, p)$ is connected, for $p = c \log n / n$ as $n \rightarrow \infty$, tends to $[1]$:

$$\lim_{n \rightarrow \infty} \mathbf{P}(G_p \in G(n, p) \text{ is connected}) = \begin{cases} 1 & \text{if } c > 1; \\ 1 - e^{-1} & \text{if } c = 1; \\ 0 & \text{if } c < 1. \end{cases} \quad (4.137)$$

Theorem 10. If Q is a convex property and $pq(\frac{n}{2}) \rightarrow \infty$, then almost every graph in $G(n, p)$ has Q iff for every fixed x a.e. graph in $G(n, M)$ has Q , when $M = \lfloor p(\frac{n}{2}) + x(pq(\frac{n}{2})^{0.5}) \rfloor$ [1].

Definition 22. Q is a convex property if $F \subset G \subset H$ and $F \in Q$ and $H \in Q$ then $G \in Q$.

We want to calculate $\mathbf{P}(\text{Per}(G_m \in G(n, M = m), p_n) \text{ is connected})$ as $n \rightarrow \infty$. From theorem 10, one can conclude that $G(n, p)$ and $G(n, M = p\binom{n}{2})$ have almost the same behavior for connectivity, as $n \rightarrow \infty$. Hence:

$$\mathbf{P}(\text{Per}(G_M \in G(n, M = p\binom{n}{2}), p_n) \text{ is connected}) \quad (4.138)$$

$$\approx \mathbf{P}(G_p \in G(n, pp_n) \text{ is connected}), \text{ as } n \rightarrow \infty \quad (4.139)$$

We can calculate $\mathbf{P}(G_p \in G(n, pp_n) \text{ is connected})$ from theorem 10 for $pp_n = c \log n / n$, as $n \rightarrow \infty$. Therefore, we should consider $p_n = c(n-1) \log n / (2m)$. Finally, one can calculate the probability that $G_M \in G(n, M = m)$ is connected after percolation with the parameter $p_n = c(n-1) \log n / (2m)$ as:

$$\lim_{n \rightarrow \infty} \mathbf{P}(\text{Per}(G_M \in G(n, M), p_n) \text{ is connected}) = \begin{cases} 1 & \text{for } p_n \text{'s such that } c > 1; \\ 1 - e^{-1} & \text{for } p_n \text{'s such that } c = 1; \\ 0 & \text{for } p_n \text{'s such that } c < 1. \end{cases} \quad (4.140)$$

Now, let us calculate $\mathbf{P}(\text{Per}(G_M \in G(2^n, n2^{n-1}), p_n) \text{ is connected})$. From (4.140), for $p_n = c(1 - \frac{1}{2^n})$, one gets:

$$\lim_{n \rightarrow \infty} \mathbf{P}(\text{Per}(G_M \in G(2^n, n2^{n-1}), p_n) \text{ is connected}) = \begin{cases} 1 & \text{for } p_n \text{'s such that } c > \log 2; \\ 1 - e^{-1} & \text{for } p_n \text{'s such that } c = \log 2; \\ 0 & \text{for } p_n \text{'s such that } c < \log 2. \end{cases} \quad (4.141)$$

Finally, it can be concluded that, $\lim_{n \rightarrow \infty} \mathbf{P}(\text{Per}(G_M \in G(2^n, n2^{n-1}), p') \text{ is connected}) = 1$ for $0.3 \approx \log 2 < p' < 0.5$, while $\lim_{n \rightarrow \infty} \mathbf{P}(Q_{p'}^n \text{ is connected}) = 0$ for $\log 2 < p' < 0.5$. Hence, a typical graph with 2^n vertices and $n2^{n-1}$ edges is "more connected" than Q^n .

CHAPTER 5

CONNECTED RANDOM SUBGRAPHS OF THE 3-CUBE

¹ The 3-cube graph, ${}^3Q^n$, is a simple graph with 3^n vertices. If one labels each vertex of ${}^3Q^n$ from 0 to $3^n - 1$, then two vertices are adjacent if their ternary representation differs only in one digit. The number of edges in ${}^3Q^n$ is $n3^n$, since each vertex is connected to $2n$ vertices. A random subgraph of ${}^3Q^n$ contains all vertices of ${}^3Q^n$, and each edge independently with probability p_n . ${}^3Q_{p_n}^n$ stands for a random subgraph of ${}^3Q^n$.

The main goal in this chapter is to explore a critical value p_c , which for fixed values of p if $p < p_c$ then the probability that ${}^3Q_{p_n}^n$ is connected, as $n \rightarrow \infty$, tends to 0; but if $p > p_c$ then the probability that ${}^3Q_{p_n}^n$ is connected, as $n \rightarrow \infty$, tends to 1. We prove that this critical value is $(\sqrt{3} - 1)/(\sqrt{3})$.

In the first section of this chapter, first the probability that ${}^3Q_{p_n}^n$ has no isolated point as $n \rightarrow \infty$, for fixed p , is investigated. It is proved that for $p < (\sqrt{3} - 1)/(\sqrt{3})$ the probability that ${}^3Q_{p_n}^n$ has no isolated point, as $n \rightarrow \infty$, tends to 0. Therefore, for $p < (\sqrt{3} - 1)/(\sqrt{3})$ the probability that ${}^3Q_{p_n}^n$ is connected, as $n \rightarrow \infty$, tends to 0. Then it is proved that, for $p > (\sqrt{3} - 1)/(\sqrt{3})$ the probability that ${}^3Q_{p_n}^n$ has no isolated point, as $n \rightarrow \infty$, tends to 1. In the next step, the probability that ${}^3Q_{p_n}^n$ has no isolated point, as $n \rightarrow \infty$, when p depends on n and it is close $(\sqrt{3} - 1)/(\sqrt{3})$, is explored. It is proved that for $\lambda(n) = \lambda > 0$ and $p_n = 1 - (1/\sqrt{3})\lambda^{1/2n}(1 + o(1/n))$, the probability that ${}^3Q_{p_n}^n$ has no isolated point, as $n \rightarrow \infty$, tends to $e^{-\lambda}$. Finally, it is proved that, for fixed $p = (\sqrt{3} - 1)/(\sqrt{3})$ the probability that ${}^3Q_{p_n}^n$ has no isolated point, as $n \rightarrow \infty$, tends to e^{-1} .

In the second section, one sheds light on the Isoperimetric problem, which is the problem of finding an inequality which relates the size of a subgraph to the size of its boundary. One needs such an inequality to explore the probability that ${}^3Q_{p_n}^n$ has a component which is not the whole graph.

In the last section, the Isoperimetric inequality is applied to prove that when p depends on n and $p_n \geq 1 - (1/\sqrt{3})(\log n)^{1/n}$, then the probability that there are no components with size larger than 1 and smaller than 3^n in ${}^3Q_{p_n}^n$, as $n \rightarrow \infty$, tends to 1. Therefore, for $p_n = 1 - (1/\sqrt{3})\lambda^{1/2n}(1 + o(1/n))$, the probability that ${}^3Q_{p_n}^n$ is connected, as $n \rightarrow \infty$, tends to $e^{-\lambda}$. Finally, as a special case, it is shown that, for fixed p if $p = (\sqrt{3} - 1)/(\sqrt{3})$, the probability that ${}^3Q_{p_n}^n$ is connected, as $n \rightarrow \infty$, tends to e^{-1} ; and if $p > (\sqrt{3} - 1)/(\sqrt{3})$, the probability that ${}^3Q_{p_n}^n$ is connected, as $n \rightarrow \infty$, tends to 1.

¹The proof presented for the 3-cube, in this chapter, is independent of the proof presented in the previous chapter; therefore, some parts of the proofs overlap each other.

5.1 Isolated vertices

For $p_n < \frac{\sqrt{3}-1}{\sqrt{3}}$:

First, one should consider the following definitions:

Definition 23. $f_n(p_n) := \mathbf{P}(^3Q_{p_n}^n \text{ is connected})$

Definition 24. $g_n(p_n) := \mathbf{P}(^3Q_{p_n}^n \text{ contains an isolated point})$

Definition 25. $X_i(n) := \begin{cases} 1 & \text{Vertex } i \text{ is isolated, } i \in V(^3Q_{p_n}^n); \\ 0 & \text{Vertex } i \text{ is NOT isolated, } i \in V(^3Q_{p_n}^n). \end{cases}$, and $X(n) := \sum_{i \in V(^3Q_{p_n}^n)} X_i(n)$.

Now, calculate $E[X(n)]$ and $Var[X(n)]$ as follows:

$$\mu := E[X(n)] = \sum_{i \in V(^3Q_{p_n}^n)} E[X_i(n)] = \sum_{i \in V(^3Q_{p_n}^n)} (1-p)^{2n} = 3^n(1-p)^{2n} \quad (5.1)$$

$$Var[X(n)] = \sum_{i \in V(^3Q_{p_n}^n)} Var[X_i(n)] + \sum_{i \neq j; i, j \in V(^3Q_{p_n}^n)} Cov[X_i(n), X_j(n)] \quad (5.2)$$

where, $Var[X_i(n)]$ and $Cov[X_i(n), X_j(n)]$ are equal to:

$$\sum_{i \in V(^3Q_{p_n}^n)} Var[X_i(n)] = 3^n(1-p)^{2n} - 3^n(1-p)^{2n}(1-p)^{2n} = \mu - \mu(1-p)^{2n} \quad (5.3)$$

$$Cov[X_i(n), X_j(n)] = E[X_i(n)X_j(n)] - E[X_i(n)]E[X_j(n)] \quad (5.4)$$

$$= \begin{cases} 0 & \text{i, j not adjacent;} \\ (1-p)^{2n}(1-p)^{2n-1} - (1-p)^{2n}(1-p)^{2n} = \frac{\mu^2}{3^{2n}}(\frac{p}{1-p}) & \text{i, j adjacent.} \end{cases} \quad (5.5)$$

and finally:

$$Var[X(n)] = \mu - \mu(1-p)^{2n} + \frac{\mu^2}{3^n}(\frac{2np}{1-p}) = \mu + \mu(1-p)^{2n}(\frac{2np}{1-p} - 1) \quad (5.6)$$

Now, since we have $Var[X(n)]$, we can use Chebyshev's inequality to estimate $g_n(p)$. Chebyshev's inequality states that:

$$1 - g_n(p) = \mathbf{P}[X(n) = 0] \leq \mathbf{P}[|X(n) - \mu| \geq \mu] \leq \frac{Var[X(n)]}{\mu^2} \quad (5.7)$$

By applying Chebyshev's inequality when $p < (\sqrt{3}-1)/(\sqrt{3})$, one gets $Var[X(n)]/\mu^2 \rightarrow 0$, as $n \rightarrow \infty$. Therefore $\lim_{n \rightarrow \infty} g_n(p) = 1$. And finally, since $f_n(p) \leq 1 - g_n(p)$, then for $p < (\sqrt{3}-1)/(\sqrt{3})$ the probability that $^3Q_{p_n}^n$ is connected, as $n \rightarrow \infty$, tends to 0. ■

For $p_n > \frac{\sqrt{3}-1}{\sqrt{3}}$:

Assume p is fixed and $p > (\sqrt{3}-1)/(\sqrt{3})$. In order to calculate $g_n(p)$ when $p > (\sqrt{3}-1)/(\sqrt{3})$, as $n \rightarrow \infty$, one can use the following inequality:

$$g_n(p) = \mathbf{P}[X(n) > 0] \leq E[X(n)] = \mu \quad (5.8)$$

Since $E[X(n)] \rightarrow 0$ as $n \rightarrow \infty$, then $\lim_{n \rightarrow \infty} g_n(p) = 0$. This means that the probability that there are no isolated points in ${}^3Q_{p_n}^n$ for $p > (\sqrt{3}-1)/(\sqrt{3})$, as $n \rightarrow \infty$, tends to 1. ■

For $p_n = 1 - (1/\sqrt{3})\lambda^{1/2n}(1 + o(1/n))$:

One needs the following theorem from [1] to find the distribution of $X(n)$ (distribution of the number of isolated points).

Theorem 11. Let $\lambda = \lambda(n)$ be a non-negative bounded function on \mathbf{N} . Suppose the non-negative integer valued random variables $X(1), X(2), \dots$ are such that:

$$\lim_{n \rightarrow \infty} \{E_r[X(n)] - \lambda^r\} = 0, \quad r = 0, 1, \dots \quad (5.9)$$

where $E_r[X]$ is the r th factorial moment of X , i.e. $E_r[X] = E[(X)_r]$. Then

$$X(n) \xrightarrow{d} \mathbf{P}_\lambda \quad (5.10)$$

Use the definition of $X(n)$ presented in definition 25. The goal is to calculate $E[X(n)]$.

$$E_r[X(n)] = E[X(n)(X(n)-1)(X(n)-2)\dots(X(n)-r+1)] \quad (5.11)$$

Since $X(n) := \sum_{i \in V(Q_p^n)} X_i(n)$ and X_i 's are indicator functions, therefore:

$$X(n)(X(n)-1)(X(n)-2)\dots(X(n)-r+1) = \sum_{(i_1, i_2, \dots, i_r)} X_{i_1} X_{i_2} \dots X_{i_r} \quad (5.12)$$

where the sum is over all ordered sets of distinct vertices. Then:

$$E_r[X(n)] = E[X(n)(X(n)-1)(X(n)-2)\dots(X(n)-r+1)] \quad (5.13)$$

$$= E\left[\sum_{(i_1, i_2, \dots, i_r)} X_{i_1} X_{i_2} \dots X_{i_r}\right] \quad (5.14)$$

$$= \sum_{(i_1, i_2, \dots, i_r)} \mathbf{P}[X_{i_1} = 1, X_{i_2} = 1, \dots, X_{i_r} = 1] \quad (5.15)$$

One knows that a set of r vertices is incident with at most $2rn$ edges. There are $(r)_r \binom{3^n}{r}$ ways to choose such r vertices. Hence:

$$E_r[X(n)] \geq (r)_r \binom{3^n}{r} (1-p_n)^{2rn} = (3^n)_r (1-p_n)^{2rn} \quad (5.16)$$

On the other hand, a set of r vertices is incident with at least $r(2n-r)$ edges. There are at most $(r-1)_{r-1} \binom{3^n}{r-1} (r-1)2n$ ways to choose a set of r vertices in ${}^3Q_{p_n}^n$ where at least two vertices are adjacent; since if we choose $r-1$ vertices independently, then the last vertex must be connected to one of the chosen vertices. In other words, there are at most $(r-1)_{r-1} \binom{3^n}{r-1} (r-1)2n$ ways to choose r vertices which some of them are adjacent to each other. Hence:

$$E_r[X(n)] \leq (3^n)_r (1-p_n)^{2rn} + (r-1)_{r-1} \binom{3^n}{r-1} 2(r-1)n(1-p_n)^{r(2n-r)} \quad (5.17)$$

$$\leq (3^n)_r (1-p_n)^{2rn} + (3^n)_r 2rn(1-p_n)^{r(2n-r)} \quad (5.18)$$

$$\leq (3^n)_r (1-p_n)^{2rn} + 3^{n(r-1)} 2rn(1-p_n)^{r(2n-r)} \quad (5.19)$$

Finally from 5.16 and 5.19 one gets:

$$(3^n)_r(1-p_n)^{2rn} \leq E_r[X(n)] \leq (3^n)_r(1-p_n)^{2rn} + 3^{n(r-1)}2rn(1-p_n)^{r(2n-r)} \quad (5.20)$$

$$(5.21)$$

which gives:

$$(3(1-p_n)^2)^{rn}(1-\frac{r}{3^n})^r \leq E_r[X(n)] \leq (3(1-p_n)^2)^{rn}\{1+3^{-n}2rn(1-p_n)^{-r^2}\} \quad (5.22)$$

Since r is fixed and $\lim_{n \rightarrow \infty} (3(1-p_n)^2)^n = \lambda$, then:

$$\lim_{n \rightarrow \infty} (E_r[X(n)]) = \lambda^r \text{ for } r=0,1,2,\dots \quad (5.23)$$

This shows that $X(n) \xrightarrow{d} \mathbf{P}_\lambda$. ■

For $p = 0.5$:

In the calculation of $p_n = 1 - 1/\sqrt{3}\lambda^{1/2n}(1+o(1/n))$, if we fix $p = (\sqrt{3}-1)/(\sqrt{3})$ and let $\lambda = 1$, then we get that the distribution of $X(n)$, as $n \rightarrow \infty$, tends to a Poisson distribution with mean 1. Therefore, one can conclude:

$$\lim_{n \rightarrow \infty} (1 - g_n(p)) = \lim_{n \rightarrow \infty} (\mathbf{P}(X(n) = 0)) = e^{-1} \quad (5.24)$$

This shows that for $p = (\sqrt{3}-1)/(\sqrt{3})$ the probability that ${}^3Q_{p_n}^n$ has no isolated point, as $n \rightarrow \infty$, tends to e^{-1} . ■

5.2 Isoperimetric problem for the 3-cube

One needs an inequality which relates the size of a subgraph of ${}^3Q^n$ to the size of its boundary. This inequality will be applied to prove that for $p \geq (\sqrt{3}-1)/(\sqrt{3})$, the probability that subgraphs of ${}^3Q^n$ do not have a component of size larger than 2 and smaller than 3^n , as $n \rightarrow \infty$, tends to 1.

Definition 26. The edge boundary $b_G(H)$, where H is an induced subgraph of G , is the number of edges which joins vertices in H to the vertices in $G \setminus H$.

The main task in this section is to calculate $b_{{}^3Q^n}(k)$. Since ${}^3Q^n$ is $2n$ -regular and H is an induced subgraph of G with $|V(H)| = m$, then:

$$b_{{}^3Q^n}(H) = mn - 2e(H), \text{ where } e(H) \text{ is the total number of edges in } H. \quad (5.25)$$

$$b_{{}^3Q^n}(k) = mn - 2e_n(k), \text{ where } e_n(k) = \max\{e(H) : H \text{ induced subgraph of } {}^3Q^n, |V(H)| = m\}. \quad (5.26)$$

Definition 27. $h(i) :=$ sum of digits in the ternary expansion of i and $f(l, m) := \sum_{l \leq i < m} h(i)$

Lemma 3. If $1 \leq k \leq l$, then $f(l, l+k) \geq f(0, k) + k$

PROOF:

Look at the ternary expansion of a few numbers:

<i>Column</i>	2	1	0
<i>Bin\Dec</i>	3^2	3^1	3^0
0			0
1			1
2			2
3		1	0
4		1	1
5		1	2
6		2	0
7		2	1
8		2	2
9	1	0	0
10	1	0	1
11	1	0	2
12	1	1	0
13	1	1	1
14	1	1	2
15	1	2	0
16	1	2	1
17	1	2	2
18	2	0	0
19	2	0	1
20	2	0	2
21	2	1	0

From this representation, one can observe that column i starts with a block of 3^i zeros. Therefore, sum of j th digits of k numbers, one after the other, is minimal if the first block of 0's is as long as possible. Hence, one can conclude:

$$f(l, l+k) \geq f(0, k) \quad (5.27)$$

For every i define r such that $0 \leq i \leq 3^r - 1$. The ternary expansions of i and $3^r - 1 - i$ are symmetric. This means that, if there is a 2/0 in an specific location of the ternary expansion of i then there is a 0/2 in the same location of the ternary expansion of $3^r - 1 - i$; and if there is a 1 in an specific location of the ternary expansion of i then there is a 1 in the same location of the ternary expansion of $3^r - 1 - i$. Therefore,

$$h(i) + h(3^r - 1 - i) = 2r \text{ for } 0 \leq i \leq 3^r - 1 \quad (5.28)$$

Consequently, since:

$$\sum_{l \leq i < l+k} h(i) + \sum_{3^r - l - k \leq i < 3^r - l} h(i) = 2rk \quad (5.29)$$

then:

$$f(l, l+k) + f(3^r - l - k, 3^r - l) = 2rk, \text{ where } l+k \leq 3^r \quad (5.30)$$

One should first prove lemma 3 with the assumption $k \leq 3^r \leq l$ by using 5.27 and ???. This assumption means that the length of the sequence of 0's, 1's and 2's in the ternary expansion

of $3^r + k$ and 3^r are equal.

With the same logic that one gets 5.27, one gets:

$$f(l, l+k) \geq f(3^r, 3^r+k), \text{ when } 3^r \leq l \quad (5.31)$$

and then for $k \leq 3^r$ one can get:

$$f(3^r, 3^r+k) = \sum_{3^r \leq i < 3^r+k} h(i) \quad (5.32)$$

$\sum_{3^r \leq i < 3^r+k} h(i)$ is the sum over numbers with the same length in their ternary expansion's sequence. When one removes the last digit in their ternary expansion, what remains is $f(0, k)$. Therefore, $\sum_{3^r \leq i < 3^r+k} h(i)$ is equal to sum of the last digits plus $f(0, k)$. Hence:

$$f(l, l+k) \geq f(3^r, 3^r+k) = k + f(0, k) \text{ when } k \leq 3^r \leq l \quad (5.33)$$

and finally:

$$f(l, l+k) \geq f(0, k) + k \text{ where } k \leq 3^r \leq l \quad (5.34)$$

Lemma 3 is proved with the assumption $k \leq 3^r \leq l$. Now, one should prove lemma 3 without this assumption. The proof is based on induction on K . We want to prove that for $1 \leq K \leq l$, $f(l+K, l) \geq K + f(0, K)$. Fix k such that $1 \leq k \leq l$ and $K < k$. For $K = 1$ the inequality is trivial. Assume that the inequality is true for $K < k$ and $K > 2$, which means:

$$f(l, l+K) \geq K + f(0, K) \text{ when } 1 \leq k \leq l, \text{ and } K < k \quad (5.35)$$

Now, one should prove the inequality for $K = k$. Define $r \geq 1$ by $3^{r-1} \leq k < 3^r$. If $l \geq 3^r$, then $k \leq 3^r \leq l$ and the lemma is implied by inequality 5.34. One may assume that $3^{r-1} < l < 3^r$. Finally, one should apply inequality 5.30 and 5.35 in order to get the final result:

$$f(l+k) = f(l, 3^r) + f(3^r, l+k) \text{ (from definition of } f \text{ and } l \geq 3^r) \quad (5.36)$$

$$= (3^r - l)2r - f(0, 3^r - l) + f(3^r, l+k) \text{ (from 5.30)} \quad (5.37)$$

$$\geq (3^r - l)2r - f(0, 3^r - l) + f(0, l+k-3^r) + l+k-3^r \text{ (from 5.35)} \quad (5.38)$$

$$\geq (3^r - l)2r - f(3^r - k, 3^r - k + 3^r - l) + 3^r - l + f(0, l+k-3^r) + l+k-3^r \text{ (from 5.35)} \quad (5.39)$$

$$\geq (3^r - l)2r - f(3^r - k, 3^r - k + 3^r - l) + f(0, l+k-3^r) + k \quad (5.40)$$

$$\geq f(l+k-3^r, k) + f(0, l+k-3^r) + k \text{ (from 5.30)} \quad (5.41)$$

$$\geq f(0, k) + k \text{ (from characteristics of } f) \quad (5.42)$$

■

Theorem 12. For $2 \leq m \leq 3^n$ we have $b_{3Q^n}(m) = mn - 2f(0, m)$. In other words, $f(0, m) = e_n(m)$ where $e_n(m) = \max\{e(H) : H \text{ induced subgraph of } 3Q^n \mid |V(H)| = m\}$.

PROOF:

First, let us fix an m . As the first step, one should prove that $e_n(m) \geq f(0, m)$. Vertex i is connected to $h(i)$ vertices j with $j < i$, since for each 1 (2) in the ternary expansion of i there is exactly one j ($j < i$), which its ternary expansion differs in the position of that 1 (2). Therefore, one can conclude that $W = \{0, 1, 2, \dots, m-1\}$ contains $\sum_{0 \leq i < m} h(i) = f(0, m)$ edges. So, $e_n(m) \geq f(0, m)$.

As the second step, one should prove that $e_n(m) \leq f(0, m)$ by induction on n for fixed m . For $n = 1$ the inequality is trivially true. Assume that it is true for $N < n$, which means:

$$e_N(m) \leq f(0, m), \text{ where } N < n \text{ and the fixed } m \text{ is } 3 \leq m \leq 3^N \quad (5.43)$$

Now, one should check the inequality 5.43 for $N = n$. This means that we should find an H induced subgraph of ${}^3Q^n$, $|V(H)| = m$, which maximize $e_n(m)$. Let us split ${}^3Q^n$ into three $(n-1)$ -dimensional cubes, face-1, face-2 and face-3 each with 3^{n-1} vertices and $(n-1)3^{n-1}$ edges. Now, one can construct H . Choose m_1 vertices for H from face-1, m_2 vertices from face-2, and m_3 vertices from face-3 where $m_1 + m_2 + m_3 = m$ and $m_1 \leq m_2 \leq m_3$ and $m_1 + m_2 \leq m_3$. In other words, H is constructed from three induced subgraphs, denoted by H_1 , H_2 , and H_3 .

Each face is a $(n-1)$ -dimensional cube, therefore inequality 5.43 holds for H_1 , H_2 and H_3 . Also, each vertex of each face is connected to exactly one vertex from one face and one vertex from the other face. Hence, the number of vertices of H is at most:

$$e_n(m) \leq f(0, m_1) + f(0, m_2) + f(0, m_3) + 2m_1 + m_2 \quad (5.44)$$

$2m_1 + m_2$ is the maximum number of edges between three faces that one can choose here. $2m_1$ is the maximum number of edges between chosen vertices in H_1 and H_2 plus the maximum number of edges between chosen vertices in H_1 and H_3 . Consequently, m_2 is the maximum number of edges between chosen vertices in H_2 and H_3 . Therefore:

$$e_n(m) \leq f(0, m_1) + f(0, m_2) + f(0, m_3) + 2m_1 + m_2 \text{ (from 5.43)} \quad (5.45)$$

$$\leq f(m_2, m_2 + m_1) + f(0, m_2) + m_1 + m_2 \text{ (from lemma 3)} \quad (5.46)$$

$$\leq f(m_3, m_3 + m_2 + m_1) + f(0, m_3) \text{ (from lemma 3)} \quad (5.47)$$

$$\leq f(0, m) \text{ (from definition of } f) \quad (5.48)$$

■

Theorem 12 shows that, if we want to choose an induced subgraph of ${}^3Q^n$, with m vertices, which has the smallest edge boundary, then we should choose the induced subgraph of ${}^3Q^n$ with the set of vertices $W = \{0, 1, 2, \dots, m-1\}$.

Corollary 2. For all k and n , $e_n(k) \leq k \lceil \log_3 k \rceil$, which is equivalent to $b_{{}^3Q^n}(k) \geq 2k(n - \lceil \log_3 k \rceil)$.

PROOF:

Let $r = \lceil \log_3 k \rceil$. Then

$$2f(0, k) = f(0, k) + f(0, k) \leq f(0, k) + f(3^r - k, 3^r) \text{ (from 5.27)} \quad (5.49)$$

$$= 2rk \text{ (from 5.30)} \quad (5.50)$$

Therefore:

$$e_n(k) = f(0, k) \leq rk = k \lceil \log_3 k \rceil \quad (5.51)$$

■

5.3 Isolated components of size larger than 2 and smaller than 3^n

Definition 28. C_s is the family of s -subsets (subsets with size s) of $V = V({}^3Q^n)$ whose induced graph is connected.

Remarks: $h(n) := o(g(n))$ means $\frac{h(n)}{g(n)} \rightarrow 0$ as $n \rightarrow \infty$.

Remarks: The following inequality will be applied a lot in the rest of this section:

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \frac{n^k}{k!} \leq \left(\frac{ne}{k}\right)^k \quad (5.52)$$

Theorem 13. If $p_n \geq 1 - \frac{1}{\sqrt{3}}(\log n)^{\frac{1}{n}}$, the probability that for some $S \in C_s, 2 \leq s \leq \frac{3^n}{2}$, no edges of ${}^3Q_{p_n}^n$ join S to $V({}^3Q^n) \setminus S$, as $n \rightarrow \infty$, tends to 0.

Note: For $\frac{3^n}{2} < s < 3^n$, if there exist a component of size smaller than 3^n then there is at least one component of size smaller than $\frac{3^n}{2}$ which contradicts with the theorem.

PROOF:

Consider $S \subset V = V({}^3Q^n)$ and set $b(S) = b_{Q^n}(H)$ where H is the induced subgraph of ${}^3Q^n$ with the set of vertices S . One can observe that:

$$\mathbf{P}(\text{No edges of } {}^3Q_{p_n}^n \text{ join } S \text{ to } V \setminus S) = (1 - p_n)^{b(S)} \quad (5.53)$$

In order to prove the theorem, it is sufficient to show:

$$\sum_{s=2}^{\frac{3^n}{2}} \sum_{S \in C_s} (1 - p_n)^{b(S)} = o(1) \quad (5.54)$$

From corollary 2, one knows that for $|S| = s$:

$$b(S) \geq b(s) \geq 2s(n - \lceil \log_3 s \rceil) \quad (5.55)$$

and therefore:

$$\sum_{S \in C_s} (1 - p_n)^{b(S)} \leq |C_s| (1 - p_n)^{b(s)} \quad (5.56)$$

One may partition $s, 2 \leq s \leq \frac{3^n}{2}$, to different intervals in order to find a small enough bound for $|C_s|$ and $(1 - p_n)^{b(s)}$.

First interval $2 \leq s \leq s_1, s_1 = \lfloor \frac{3^{\frac{n}{2}}}{n^2} \rfloor$:

First, one should find a bound for $|C_s|$. One has maximum 3^n choices to choose the first element for C_s . The selected element is connected to maximum $2n$ vertices, therefore, there are maximum $2n$ choices to choose the second element. With the same logic, there are at most $2n(s-1)$ choices to choose the last element for $|C_s|$. Hence, one can show:

$$|C_s| \leq 3^n(2n)(2n(2))\dots(2n(s-1)) \leq (s-1)!(2n)^{s-1}3^n \quad (5.57)$$

and:

$$|C_s|(1 - p_n)^{b(s)} \leq (s-1)!(2n)^{s-1}3^n(1 - p_n)^{2s(n - \lceil \log_3 s \rceil)} \quad (5.58)$$

Since $p_n = 1 - \frac{1}{\sqrt{3}}(\log n)^{\frac{1}{n}}$, so for large enough n :

$$(1 - p_n)^{2s(n - \lceil \log_3 s \rceil)} \leq (3)^{-ns}(\log n)^{2s}(1 - p_n)^{-2s(\log_3 s)} (\text{neglecting some small terms}) \quad (5.59)$$

$$(\text{since: } (\log n)^{\frac{-2s \log_3 s}{n}} \leq 1 \text{ for large enough } n) \quad (5.60)$$

$$= (3)^{-ns}(\log n)^{2s}3^{s \log_3 s}(\log n)^{\frac{-2s \log_3 s}{n}} \quad (5.61)$$

$$\leq (3)^{-ns}(\log n)^{2s}s^s \quad (5.62)$$

From equations 5.58 and 5.62, one gets:

$$|C_s|(1-p_n)^{b(s)} \leq (s-1)!(2n)^{s-1}3^n(3)^{-ns}(\log n)^{2s}s^s \quad (5.63)$$

Assume that the right hand sides of inequality 5.63 is equal to A. After multiplying both side of inequality 5.63 with $\frac{2ns^{s+1}}{s!}$ and then getting \log_3 from both sides, one gets:

$$\log_3(|C_s|(1-p_n)^{b(s)}\frac{2ns^{s+1}}{s!}) \leq \log_3(A\frac{2ns^{s+1}}{s!}) \quad (5.64)$$

If $\log_3(A\frac{2ns^{s+1}}{s!}) \rightarrow -\infty$ as $n \rightarrow \infty$ then $A\frac{2ns^{s+1}}{s!}$ should tend to 0. This means that $|C_s|(1-p_n)^{b(s)}\frac{2ns^{s+1}}{s!}$ tends to 0, as $n \rightarrow \infty$. Therefore:

$$|C_s|(1-p_n)^{b(s)} \leq \frac{s!}{2ns^{s+1}} \text{ for large values of } n \quad (5.65)$$

which shows that:

$$\sum_{s=2}^{s_1} \sum_{S \in C_s} (1-p_n)^{b(S)} = o(1) \quad (5.66)$$

Finally, it remains to prove $\log_3(A\frac{2ns^{s+1}}{s!}) \rightarrow -\infty$ as $n \rightarrow \infty$. One can verify this for $s \leq n$ and $s > n$. ■

Second interval $s_1 + 1 \leq s \leq \frac{3^n}{2}$ and $S \in C_s^-, s_1 = \lfloor \frac{3^{\frac{n}{2}}}{n^2} \rfloor$:
Define C_s^- and C_s^+ as follows:

Definition 29.

$$C_s^- := \{S \in C_s | b(s) \geq 2s(n - \log_3 s + \log_3 n)\}, \text{ and } C_s^+ := C_s \setminus C_s^- \quad (5.67)$$

One can bound $|C_s^-|$ for $s_1 + 1 \leq s \leq 3^{\frac{n}{2}}$ as follows:

$$|C_s^-| \leq |C_s| \leq \binom{3^n}{s} \leq \frac{3^{ns}}{s!} \leq \left(\frac{e3^n}{s}\right)^s \quad (5.68)$$

Hence:

$$\sum_{s=s_1+1}^{3^{\frac{n}{2}}} \sum_{S \in C_s^-} (1-p_n)^{b(S)} \leq \sum_{s=s_1+1}^{3^{\frac{n}{2}}} \left(\frac{e3^n}{s}\right)^s (\sqrt{3}^{-1}(\log n)^{\frac{1}{n}})^{2s(n-\log_3 s+\log_3 n)} \quad (5.69)$$

$$\leq \sum_{s=s_1+1}^{3^{\frac{n}{2}}} \left(\frac{e3^n 3^{-(n-\log_3 s+\log_3 n)} (\log n)^{\frac{2(n-\log_3 s+\log_3 n)}{n}}}{s}\right)^s \quad (5.70)$$

$$\leq \sum_{s=s_1+1}^{3^{\frac{n}{2}}} \left(\frac{e3^n 3^{-n} 3^{\log_3 s} 3^{-\log_3 n} (\log n)^2}{s}\right)^s (\log n)^{\frac{2n(-\log_3 s+\log_3 n)}{s}} \quad (5.71)$$

$$(\text{since for large enough } n: (\log n)^{\frac{2n(-\log_3 s+\log_3 n)}{s}} \leq 1) \quad (5.72)$$

$$\leq \sum_{s=s_1+1}^{3^{\frac{n}{2}}} \left(\frac{e(\log n)^2}{n}\right)^s = o(1) \quad (5.73)$$

■

Third interval $s_1 \leq s \leq s_2, s_1 = \lfloor \frac{3^{\frac{n}{2}}}{n^2} \rfloor, s_2 = \lfloor \frac{3^n}{(\log n)^2} \rfloor$ and $S \in C_s^+$:

For the 3rd and the 4th intervals one needs to find a bound for $|C_s^+|$. The following lemma, presented by B.Bollobas [1], helps us in this matter:

Lemma 4. Let G be a graph of order v and suppose that $\Delta(G) \leq \Delta$, $2e(G) = vd$ and $\Delta + 1 \leq u \leq v - \Delta - 1$. Then, there is a u -set of U of vertices with [1]:

$$|N(U)| = |U \cup \Gamma(U)| \geq v \frac{d}{\Delta} \{1 - \exp(\frac{-u(\Delta + 1)}{v})\} \quad (5.74)$$

where, $\Delta(G) :=$ Maximum degree in G , $d :=$ average degree in G and $\Gamma(U) = \{x \in V(G) : xy \in E(G) \text{ for some } y \in U\}$

Let $H = {}^3Q^n[S]$ (the induced subgraph of 3Q_n with the set of vertices S). From the definition of C_s^+ one knows that the average degree in H is at least:

$$2(\log_3 s - \log_3 n) \quad (5.75)$$

The goal is to find $U \subset S$, where $|U| := u := \lfloor \frac{2s}{n} \rfloor$, $\Delta = 2n$, $v = s$, $d \geq \log_3 s - \log_3 n$ and then use lemma 4 to calculate the boundary size of U , $|N(U)|$. First, check if $2n + 1 \leq \lfloor \frac{2s}{n} \rfloor$, as $n \rightarrow \infty$:

$$\frac{2s}{n} = \frac{23^{n/2}}{n^3} \text{ for minimum } s, \text{ and trivially } 2n + 1 \leq \frac{23^{n/2}}{n^3}, \text{ for large enough } n \quad (5.76)$$

$$(5.77)$$

and then check if $\lfloor \frac{2s}{n} \rfloor \leq s - (2n + 1)$. One should check if $ns - n(2n + 1) \geq 2s$ which means one should check that whether:

$$\frac{3^{\frac{n}{2}}(n - 2)}{n^3(2n + 1)} \geq 1 \quad (5.78)$$

which is clearly true for large enough n . Now, one can apply lemma 4 on the graph generated by S and get:

$$\exists U \subset S : |N(U)| \geq s \frac{2(\log_3 s - \log_3 n)}{2n} \{1 - \exp(-\frac{u(2n + 1)}{s})\} \quad (5.79)$$

where:

$$\frac{2(\log_3 s - \log_3 n)}{2n} \geq \frac{(\log_3(\frac{3^{\frac{n}{2}}}{n^2}) - \log_3 n)}{n} = \frac{\frac{n}{2} - 3 \log_3 n}{n} \quad (5.80)$$

$$\text{which } \lim_{n \rightarrow \infty} \frac{\frac{n}{2} - 3 \log_3 n}{n} = \frac{1}{2} \quad (5.81)$$

on the other hand:

$$\lim_{n \rightarrow \infty} (1 - \exp(-\frac{2n + 1}{s}(\frac{2s}{n} + 1))) = 1 - e^{-4} \quad (5.82)$$

Therefore, from 5.81 and 5.82 one gets:

$$|N(U)| \geq \frac{1}{2}(1 - e^{-4})s \geq \frac{s}{3} \text{ as } n \rightarrow \infty \quad (5.83)$$

Now that we have $|N(U)|$, we can estimate a bound for $|C_s^+|$ here. We know from 5.83 that for each $S \in C_s^+$ there exist a $U \subseteq S$, $|U| := u := \lfloor \frac{2s}{3} \rfloor$, such that $|N(U)| \geq s/3$. Therefore, one can choose $S \in C_s^+$ as follows:

1. Select u vertices of ${}^3Q^n$; there are $\binom{3^n}{u}$ choices for this u .
2. Select $\lfloor \frac{s}{3} \rfloor - u$ neighbors of the selected vertices of u in part 1; there are maximum $(2^n)^u$ choices, since there are at most $\binom{2n}{0} + \binom{2n}{1} + \binom{2n}{2} + \dots + \binom{2n}{2n} = 2^{2n}$ ways to find neighbors of a vertex in U .
3. Select $\lfloor \frac{2s}{3} \rfloor$ other vertices; there are at most $\binom{3^n}{\lfloor \frac{2s}{3} \rfloor}$ choices.

Hence:

$$|C_s^+| \leq \binom{3^n}{u} (2^{2n})^u \binom{3^n}{\lfloor \frac{2s}{3} \rfloor} \quad (5.84)$$

and:

$$\sum_{S \in C_s^+} (1 - p_n)^{b(S)} \leq \binom{3^n}{u} (2^{2n})^u \binom{3^n}{\lfloor \frac{2s}{3} \rfloor} (1 - p_n)^{b(s)} \quad (5.85)$$

where:

$$(1 - p_n)^{b(s)} \leq 3^{-sn} s^s (\log n)^{2s} \quad (5.86)$$

consequently from 5.85, 5.86 and 5.52:

$$\sum_{S \in C_s^+} (1 - p_n)^{b(S)} \leq \left(\frac{e3^n}{u}\right)^u 2^{2un} \left(\frac{e3^n}{\lfloor \frac{2s}{3} \rfloor}\right)^{\lfloor \frac{2s}{3} \rfloor} 3^{-sn} s^s (\log n)^{2s} \quad (5.87)$$

Write $s = 3^{\beta n}$, ($\beta = \frac{\log_3 s}{n}$), so that:

$$3^{\beta n} \leq \frac{3^n}{(\log n)^7} \Rightarrow \beta \leq 1 - \frac{7 \log_3 \log n}{n} \quad (5.88)$$

Now, find a bound for the inequality 5.87. First calculate the first part of the inequality:

$$\left(\frac{e3^n}{u}\right)^u 2^{2un} \left(\frac{e3^n}{\lfloor \frac{2s}{3} \rfloor}\right)^{\lfloor \frac{2s}{3} \rfloor} \leq \left(\frac{e3^n}{\frac{2s}{n}}\right)^{\frac{2s}{n}} 3^{2s} 2^{4s} \left(\frac{e3^n}{\frac{2s}{3}}\right)^{\frac{2s}{3}} \quad (5.89)$$

$$\left(\text{since for large enough } n \text{ and } s_1 \leq s \leq s_2: \left(\frac{e3^n}{\frac{2s}{n}}\right)^{\frac{2s}{n}} \leq 1\right) \quad (5.90)$$

$$\leq (3^2 2^4 \left(\frac{3}{2} e\right)^{\frac{2}{3}})^s \frac{3^{\frac{2s}{3}}}{s^{\frac{2s}{3}}} = c^s 3^{\frac{2}{3} sn (1 - \frac{\log_3 s}{n})} = c^s 3^{\frac{2}{3} sn (1 - \beta)} \quad (5.91)$$

where c is a positive constant. Now, by substituting 5.91 in 5.87 one gets:

$$\sum_{S \in C_s^+} (1 - p_n)^{b(S)} \leq 3^{-sn} s^s (\log n)^{2s} c^s 3^{\frac{2}{3} sn (1 - \beta)} \quad (5.92)$$

$$= c^s (\log n)^{2s} 3^{-\frac{sn(1-\beta)}{3}} \quad (5.93)$$

$$\leq c^s (\log n)^{2s} 3^{-\frac{7s \log_3 \log n}{3n}}, \text{ (from 5.88)} \quad (5.94)$$

$$= c^s (\log n)^{2s} 3^{\log_3 (\log n) \frac{-7s}{3}} \quad (5.95)$$

$$\leq c^s (\log n)^{2s} (\log n)^{\frac{-7s}{3}} \quad (5.96)$$

$$= c^s (\log n)^{\frac{-s}{3}} \quad (5.97)$$

and finally from 5.97:

$$\sum_{s=s_1}^{s_2} \sum_{S \in C_s^+} (1-p_n)^{b(S)} \leq \sum_{s=s_1}^{s_2} c^s (\log n)^{\frac{-s}{3}} = o(1) \quad (5.98)$$

■

Fourth interval $s_2 + 1 \leq s \leq \frac{3^n}{2}$ and $s_2 = \lfloor \frac{3^n}{(\log n)^9} \rfloor, S \in C_s^+$:

In $H = {}^3Q^n[S]$ (the induced subgraph ${}^3Q^n$ with the set of vertices S), the average degree is at least:

$$2(\log_3 s - \log_3 n) > 2(n - 2 \log_3 n) \quad (5.99)$$

since:

$$s \geq \lceil \frac{3^n}{(\log n)^9} \rceil \Rightarrow \log_3(\frac{3^n}{(\log n)^9}) < \log_3 s \quad (5.100)$$

$$\Rightarrow \log_3 s - \log_3 n \geq \log_3(\frac{3^n}{(\log n)^9}) - \log_3 n \geq n - \log_3(\log n)^9 - \log_3 n \quad (5.101)$$

$$(\text{for large enough } n \text{ one can get } n > (\log n)^9) \quad (5.102)$$

$$\geq n - 2 \log_3 n \quad (5.103)$$

First, look for a subgraph of H with large average degree. Let T be the set of vertices of H with degree at least $2(n - (\log_3 n)^2)$, and set $t = |T|$. From 5.99 one can conclude that the sum of degrees in H is at least $s(n - 2 \log_3 n)$. We also know that:

$$\text{Sum of degrees in } S \leq 2s(n - 2 \log_3 n) \quad (5.104)$$

$$\leq t \times (\text{Maximum degree of vertices in set } T \text{ of graph } H) \quad (5.105)$$

$$+ (s - t) \times (\text{Maximum degree of vertices in set } S \setminus T \text{ of graph } H) \quad (5.106)$$

$$\leq 2tn + 2(s - t)(n - (\log_3 n)^2) \quad (5.107)$$

$$\Rightarrow t \geq s(1 - \frac{4}{\log_3 n}) \quad (5.108)$$

Define $H_1 = {}^3Q^n[T] = H[T]$ as the induced subgraph spanned by T . We want to calculate $|N_{H_1}(U)|$ in H_1 , hence we should estimate the size of H_1 and after that calculate the average degree in T . Let us first calculate $e(H_1)$, the total number of edges in H_1 .

$$e(H_1) \geq e(H) - 2(s - t)n \geq 2\frac{s}{2}(n - 2 \log_3 n) - \frac{4s}{\log_3 n}n \text{ (from 5.99 and 5.108)} \quad (5.109)$$

One knows that the average degree in H_1 is at least $\frac{2e(H_1)}{s}$, and:

$$\frac{2e(H_1)}{s} \geq 2(n - 2 \log_3 n) - \frac{8n}{\log_3 n} \geq 2n - \frac{9}{\log_3 n} \quad (5.110)$$

$$(\text{since: } \log_3 n^2 < \frac{n}{\log_3 n} \text{ for large enough } n) \quad (5.111)$$

Set $u = \lfloor \frac{3^n}{n^2} \rfloor$. One should check the conditions of lemma 3 here. Let $v = t, \Delta = 2n, d \geq 2n - \frac{9}{\log_3 n}$. So, one should check if $2n + 1 \leq \frac{3^n}{n^2} \leq t - 2(n + 1)$, for large enough n . Clearly,

$2n + 1 \leq \frac{3^n}{n^{\frac{1}{2}}}$, as $n \rightarrow \infty$. It remains to prove $\frac{3^n}{n^{\frac{1}{2}}} \leq t - 2(n + 1)$, for large enough n . For minimum s from 5.108 we can get:

$$t \geq \frac{3^n}{(\log n)^7} \left(1 - \frac{4}{\log_3 n}\right) \quad (\text{from 5.108}) \quad (5.112)$$

$$\geq \frac{3^n}{n^{\frac{1}{2}}} + 2(n + 1) \quad (\text{for large enough } n) \quad (5.113)$$

Now, one can use lemma 3 and estimate $|N_{H_1}(U)|$.

$$|N_{H_1}(U)| \geq \frac{t}{2n} \left(2n - \frac{9n}{\log_3 n}\right) \left\{1 - \exp\left(-\frac{2n+1}{t} \frac{3^n}{n^{\frac{1}{2}}}\right)\right\} \quad (5.114)$$

$$\geq \frac{t}{2} \left(n - \frac{9}{\log_3 n}\right) \left\{1 - \exp\left(-\frac{2n+1}{t} \frac{3^n}{n^{\frac{1}{2}}}\right)\right\} \quad (5.115)$$

Let us estimate a bound for $\exp\left(-\frac{2n+1}{t} \frac{3^n}{n^{\frac{1}{2}}}\right)$. One knows that $t \geq s\left(1 - \frac{4}{\log_3 n}\right)$. Since $\max(t) = s$ and $\max(s) = \frac{3^n}{2}$, then:

$$\frac{3^n(2n+1)}{n^{\frac{1}{2}}t} \geq \frac{3^n(2n+1)}{n^{\frac{1}{2}}3^n} = \frac{(2n+1)}{n^{\frac{1}{2}}} \geq n^{\frac{1}{4}} \quad (\text{for large enough } n) \quad (5.116)$$

$$\Rightarrow \left\{1 - \exp\left(-\frac{2n+1}{t} \frac{3^n}{n^{\frac{1}{2}}}\right)\right\} \geq \exp(-n^{\frac{1}{4}}) \quad (\text{for large enough } n) \quad (5.117)$$

By using the bound from 5.117 in 5.115, one gets:

$$|N_H(U)| \geq |N(H_1)| \geq \frac{t}{2} \left(2 - \frac{9}{\log_3 n}\right) \left\{1 - \exp(-n^{\frac{1}{4}})\right\} \quad (5.118)$$

$$= \frac{t}{2} \left\{2 + \exp(-n^{\frac{1}{4}}) \frac{9}{\log_3 n} - 2 \exp(-n^{\frac{1}{4}}) - \frac{9}{\log_3 n}\right\} \quad (5.119)$$

$$\left(\lim_{n \rightarrow \infty} \exp(-n^{\frac{1}{4}}) \frac{9}{\log_3 n} = 0 \text{ and } \exp(-n^{\frac{1}{4}}) < \frac{1}{\log_3 n} \text{ (for large enough } n) \right) \quad (5.120)$$

$$\geq \frac{t}{2} \left\{2 - \frac{2}{\log_3 n} - \frac{9}{\log_3 n}\right\} = \frac{t}{2} \left(2 - \frac{11}{\log_3 n}\right) \quad (5.121)$$

$$\geq \frac{s}{2} \left(1 - \frac{2}{\log_3 n}\right) \left(2 - \frac{11}{\log_3 n}\right) = \frac{s}{2} \left(2 + \frac{2}{\log_3 n} \frac{11}{\log_3 n} - \frac{15}{\log_3 n}\right) \quad (\text{for large enough } n) \quad (5.122)$$

$$\geq \frac{s}{2} \left(2 - \frac{15}{\log_3 n}\right) = s \left(1 - \frac{7.5}{\log_3 n}\right) \quad (5.123)$$

Now that we have $|N_H(U)|$, we can estimate a bound for $|C_s^+|$ here. We know from 5.123 that for each $S \in C_s^+$ there exist a $U \subseteq S$, $|U| := u := \lfloor \frac{3^n}{n^{\frac{1}{2}}} \rfloor$, such that $|N_H(U)| \geq s \left(1 - \frac{7.5}{\log_3 n}\right)$.

Therefore, one can choose $S \in C_s^+$ as follows:

1. Select u vertices of ${}^3Q^n$; there are $\binom{3^n}{u}$ choices for this u .
2. Select $\lfloor s \left(1 - \frac{7.5}{\log_3 n}\right) \rfloor - u$ neighbors of the selected vertices in part 1. At most $2(\log_3 n)^2$ of the $2n$ neighbors of a vertex in U do not belong to $N_H(U)$. Hence there are at most $\sum_{(k_j)} \left(\prod_{i=1}^u \binom{2n}{j}\right)$ ways to find neighbors of u vertices in U , where the sum is over all (k_1, k_2, \dots, k_u) , $k_i \leq$

$2(\log_3 n)^2$. We know that:

$$\sum_{(k_i)} \prod_{i=1}^u \binom{2n}{k_i} \leq \sum_{(k_i)} \prod_{i=1}^u \left(\frac{(2n)^i}{i!} \right) \quad (5.124)$$

$$\leq \sum_{(k_i)} \prod_{i=1}^u \left(\frac{(2n)^{(2(\log_3 n)^2)}}{(2(\log_3 n)^2)!} \right) \quad (5.125)$$

$$= \sum_{(k_i)} \frac{(2n)^{2u(\log_3 n)^2}}{((2(\log_3 n)^2)!)^u} \quad (5.126)$$

$$= (2(\log_3 n)^2)^u \frac{(2n)^{2u(\log_3 n)^2}}{((2(\log_3 n)^2)!)^u} \quad (5.127)$$

$$\leq (2n)^{2u(\log_3 n)^2} \quad (5.128)$$

$$(5.129)$$

3. Select $\lfloor \frac{7.5}{\log_3 n} \rfloor$ other vertices; there are $\binom{3^n}{\lfloor \frac{7.5}{\log_3 n} \rfloor}$ choices.

Hence:

$$\sum_{S \in C_s^+} (1 - p_n)^{b(S)} \leq \binom{3^n}{u} (2n)^{2u(\log_3 n)^2} \binom{3^n}{\lfloor \frac{7.5}{\log_3 n} \rfloor} 3^{-2s(n - \log_3 s)} (\log n)^{2s(1 - \frac{\log_3 s}{n})} \quad (5.130)$$

$$(5.131)$$

where:

$$\binom{3^n}{u} (2n)^{2u(\log_3 n)^2} \binom{3^n}{\lfloor \frac{7.5}{\log_3 n} \rfloor} = 3^{o(s)} \quad (5.132)$$

Therefore:

$$\sum_{S \in C_s^+} (1 - p_n)^{b(S)} \leq 3^{\varepsilon(s)} \quad (5.133)$$

where:

$$\varepsilon(s) = o(s) - 2s\{n - \log_3 s - \log_3 \log n + \frac{\log_3 s}{n} \log_3 \log n\} \quad (5.134)$$

Since $s \leq \frac{3^n}{2}$, hence one can get:

$$\varepsilon(s) \leq o(s) - 2s\{n - (n - 1) - \log_3 \log n + \frac{n - 1}{n} \log_3 \log n\} \quad (5.135)$$

$$= o(s) - 2s\{1 - \frac{1}{n} \log_3 \log n\} \leq -s \quad (5.136)$$

Therefore, for large enough n, one can get:

$$\sum_{s=\frac{3^n}{2}+1}^{\frac{3^n}{2}} \sum_{S \in C_s^+} (1 - p_n)^{b(S)} \leq \sum_{s=\frac{3^n}{2}+1}^{\frac{3^n}{2}} 3^{-s} = o(1) \quad (5.137)$$

■

CHAPTER 6

CONNECTED RANDOM SUBGRAPH OF THE P_3 -PRODUCT

The P_3^n graph is the cartesian products of n copies of P_3 , where P_3 stands for a path with length 2. If one labels each vertex of P_3^n from 0 to $3^n - 1$, then two vertices are adjacent if the difference between their ternary representation is 1, in other words, vertex $x = (x_1, x_2, \dots, x_n)$ is connected to the vertex $y = (y_1, y_2, \dots, y_n)$ if for some i we have $|x_i - y_i| = 1$ and $x_j = y_j$ for $j \neq i$. Vertex $x = (x_1, x_2, \dots, x_n)$ is connected to $n + i$ vertices where $i = |\{j | x_j = 1, j \in \{1, \dots, n\}\}|$. A random subgraph of P_3^n contains all vertices of P_3^n , and each edge independently with probability p . p is called the percolation parameter and $P_{3,p}^n$ stands for the random subgraph of P_3^n .

The main goal in this chapter is to explore a critical value p_c , which for fixed values of p if $p < p_c$ then almost no random subgraphs of P_3^n is connected, as $n \rightarrow \infty$; but if $p > p_c$ then almost all random subgraphs of P_3^n are connected, as $n \rightarrow \infty$. We suggest that this critical value is $2 - \sqrt{2}$. The proof in this chapter is not complete and there is a place for further work.

In the first section of this chapter, it is proved that for $p < 2 - \sqrt{2}$ almost no random subgraphs of P_3^n are connected, as $n \rightarrow \infty$. Then, it is proved that, for $p > 2 - \sqrt{2}$ almost all random subgraphs of P_3^n have no isolated point, as $n \rightarrow \infty$. In the second section, the probability that random subgraphs of P_3^n , with the percolation parameter $p > 2 - \sqrt{2}$, have no components with size larger than 1 and smaller than 3^n , as $n \rightarrow \infty$, is explored.

6.1 Isolated vertices

For $p < 2 - \sqrt{2}$:

Let us define X_i and X for a graph G as follows:

Definition 30. $X_i(n) := \begin{cases} 1 & \text{Vertex } i \text{ is isolated, } i \in V(G); \\ 0 & \text{Vertex } i \text{ is NOT isolated, } i \in V(G). \end{cases}$, and $X(n) := \sum_{i \in V(G)} X_i(n)$.

As the first step we calculate $E[X(n)]$. One can categorize the set of vertices V into $n + 1$ subsets V_i 's where $x = (x_1, x_2, \dots, x_n) \in V_i$ if $|\{j | x_j = 1, j \in \{1, \dots, n\}\}| = i$. Hence:

$$\mu := E[X(n)] = \sum_{j \in V(P_{3,p}^n)} E[X_j(n)] = \sum_{i=0}^n \sum_{k \in V_i} E[X_k(n)] = \sum_{i=0}^n \binom{n}{i} 2^{n-i} (1-p)^{n+i} = (1-p)^n (3-p)^n \quad (6.1)$$

From (6.1), the threshold value for $E[X]$ is $p_c = 2 - \sqrt{2}$ which is the solution to the equation $(1-p)^n(3-p)^n = 1$. This means that $E[X] \rightarrow \infty$ for $p < p_c$, but $E[X] \rightarrow 0$ for $p > p_c$. Now, one should calculate $Var[X]$.

$$Var[X(n)] = \sum_{i \in V(P_{3,p}^n)} Var[X_i(n)] + \sum_{i,j \in V(P_{3,p}^n), i \neq j} Cov[X_i(n), X_j(n)] \quad (6.2)$$

$$= \sum_{i=0}^n \sum_{k \in V_i} Var[X_k(n)] + \sum_{i,j \in V(P_{3,p}^n), i \neq j} Cov[X_i(n), X_j(n)] \quad (6.3)$$

where:

$$\sum_{i=0}^n \sum_{k \in V_i} Var[X_k(n)] = \sum_{i=0}^n \binom{n}{i} 2^{n-i} ((1-p)^{n+i} - (1-p)^{2n+2i}) \quad (6.4)$$

$$= (1-p)^n (3-p)^n - (1-p)^{2n} (2 + (1-p)^2)^n \quad (6.5)$$

In order to calculate $\sum_{i,j \in V(P_{3,p}^n), i \neq j} Cov[X_i(n), X_j(n)]$ one should know that if $x \in V_i$ then x is incident to $2i$ vertices in V_{i-1} and $n-i$ vertices in V_{i+1} . Also, one knows:

$$Cov[X_i(n), X_j(n)] = E[X_i(n)X_j(n)] - E[X_i(n)]E[X_j(n)] = 0 \text{ if } i, j \text{ not adjacent} \quad (6.6)$$

Hence:

$$\sum_{i,j \in V(P_{3,p}^n), i \neq j} Cov[X_i(n), X_j(n)] \quad (6.7)$$

$$= \sum_{i=0}^n \binom{n}{i} 2^{n-i} \{ (n-i)(1-p)^{n+i-1} (1-p)^{n+i+1} + 2i(1-p)^{n+i-1} (1-p)^{n+i-1} \} \quad (6.8)$$

$$= \sum_{i=0}^n \binom{n}{i} 2^{n-i} (1-p)^{2n+2i-1} \{ (n-i)(1-p) + \frac{2i}{1-p} \} \quad (6.9)$$

$$= (1-p)^{2n} \sum_{i=0}^n \binom{n}{i} 2^{n-1} (1-p)^{2i-1} \{ n(1-p) + i(\frac{2}{1-p} - (1-p)) \} \quad (6.10)$$

$$= n(1-p)^{2n} \sum_{i=0}^n \binom{n}{i} 2^{n-i} (1-p)^{2i} + (1-p)^{2n-2} (2 - (1-p)^2) \sum_{i=0}^n \binom{n}{i} 2^{n-i} (1-p)^{2i} \quad (6.11)$$

$$= n(1-p)^{2n} (2 + (1-p)^2)^n + (1-p)^{2n-2} (1 - (1-p)^2) 2^n n \frac{(1-p)^2}{2} (1 + \frac{(1-p)^2}{2})^{n-1} \quad (6.12)$$

$$= n(1-p)^{2n} (2 + (1-p)^2)^n + n(2 - (1-p)^2) (2 + (1-p)^2)^{n-1} (1-p)^{2n} \quad (6.13)$$

$$= 4n(1-p)^{2n} (2 + (1-p)^2)^{n-1} \quad (6.14)$$

and finally, from (6.5) and (6.14) one gets:

$$Var[X(n)] = (1-p)^n (3-p)^n - (1-p)^{2n} (2 + (1-p)^2)^n + 4n(1-p)^{2n} (2 + (1-p)^2)^{n-1} \quad (6.15)$$

$$= \mu - \mu^2 \frac{(2 + (1-p)^2)^n}{(3-p)^{2n}} + \mu^2 \frac{4n(2 + (1-p)^2)^n}{(3-p)^{2n}} \quad (6.16)$$

Now, since we have $\text{Var}[X(n)]$, we can use Chebyshev's inequality to estimate $g_n(p)$. Chebyshev's inequality states that:

$$1 - \mathbf{P}[P_{3,p}^n \text{ contains an isolated point}] = \mathbf{P}[X(n) = 0] \leq \mathbf{P}[|X(n) - \mu| \geq \mu] \leq \frac{\text{Var}[X(n)]}{\mu^2} \quad (6.17)$$

From 6.16 when $p < 2 - \sqrt{2}$, one gets $\text{Var}[X(n)]/\mu^2 \rightarrow 0$, as $n \rightarrow \infty$. Therefore, from (6.17) one gets $\lim_{n \rightarrow \infty} \mathbf{P}[P_{3,p}^n \text{ contains an isolated point}] = 1$. Finally, since

$$\mathbf{P}[P_{3,p}^n \text{ is connected}] \leq \mathbf{P}[P_{3,p}^n \text{ does not contain an isolated point}]$$

, then for $p < 2 - \sqrt{2}$ the probability that $P_{3,p}^n$ is connected, as $n \rightarrow \infty$, tends to 0. \blacksquare

For $p > 2 - \sqrt{2}$:

In order to calculate $\mathbf{P}[P_{3,p}^n \text{ contains an isolated point}]$ when $p > 2 - \sqrt{2}$, as $n \rightarrow \infty$, one can use the following inequality:

$$\mathbf{P}[P_{3,p}^n \text{ contains an isolated point}] = \mathbf{P}[X(n) > 0] \leq E[X(n)] = \mu \quad (6.18)$$

Since $E[X(n)] \rightarrow 0$ as $n \rightarrow \infty$, then $\lim_{n \rightarrow \infty} \mathbf{P}[P_{3,p}^n \text{ contains an isolated point}] = 0$. This means that the probability that there are no isolated points in $P_{3,p}^n$ for $p > 2 - \sqrt{2}$, as $n \rightarrow \infty$, tends to 1. \blacksquare

6.2 Isolated components of size larger than 2 and smaller than 3^n

Definition 31. C_s is the family of subsets of $V(P_3^n)$ with size s whose induced graph is connected.

Definition 32. The edge boundary $b_G(H)$, where H is an induced subgraph of G , is the number of edges which joins vertices in H to the vertices in $G \setminus H$. Then $b_G(k) = \min\{b_G(H) : H \subset G, |H| = k\}$

Theorem 14. If $p \geq 2 - \sqrt{2}$, the probability that for some $S \in C_s$, $2 \leq s \leq 3^{0.44n}$, no edges of $P_{3,p}^n$ join S to $V(P_3^n) \setminus S$, as $n \rightarrow \infty$, tends to 0.

PROOF:

Consider $S \subset V = V(P_3^n)$ and set $b(S) = b_{P_3^n}(H)$ where H is the induced subgraph of P_3^n with the set of vertices S . One can observe that:

$$\mathbf{P}(\text{No edges of } P_3^n \text{ join } S \text{ to } V \setminus S) = (1 - p)^{b(S)} \quad (6.19)$$

In order to prove the theorem, it is sufficient to show:

$$\sum_{s=2}^{3^{0.44n}} \sum_{S \in C_s} (1 - p)^{b(S)} = o(1) \quad (6.20)$$

From [16] one knows that for $|S| = s$:

$$b(S) \geq b(s) \geq \frac{e}{3}s \ln \frac{3^n}{s} = \frac{e \ln 3}{3}s(n - \log_3 s) \quad (6.21)$$

one knows that:

$$\sum_{S \in C_s} (1-p)^{b(S)} \leq |C_s|(1-p)^{b(s)} \quad (6.22)$$

First, one should find a bound for $|C_s|$. One has maximum 3^n choices to choose the first element for C_s . The selected element is connected to maximum $2n$ vertices, therefore, there are maximum $2n$ choices to choose the second element. With the same logic, there are at most $2n(s-1)$ choices to choose the last element for $|C_s|$. Hence, one can show:

$$|C_s| \leq 3^n(2n)(2n(2))...(2n(s-1)) \leq (s-1)!(2n)^{s-1}3^n \quad (6.23)$$

and:

$$|C_s|(1-p)^{b(s)} \leq (s-1)!(2n)^{s-1}3^n(1-p)^{\frac{e \ln 3}{3}s(n-\log_3 s)} \quad (6.24)$$

Set $a := \frac{e \ln 3}{3}$. Assume that the right hand sides of inequality (6.24) is equal to A. After multiplying both sides of inequality (6.24) with $\frac{2ns^{s+1}}{s!}$ and then taking \log_3 from both sides, one gets:

$$\log_3(|C_s|(1-p)^{b(s)} \frac{2ns^{s+1}}{s!}) \leq \log_3(A \frac{2ns^{s+1}}{s!}) \quad (6.25)$$

If $\log_3(A \frac{2ns^{s+1}}{s!}) \rightarrow -\infty$ as $n \rightarrow \infty$ then $A \frac{2ns^{s+1}}{s!}$ should tend to 0 for $2 \leq s \leq 3^{0.44n}$. This means that $|C_s|(1-p)^{b(s)} \frac{2ns^{s+1}}{s!}$ tends to 0 for $2 \leq s \leq 3^{0.44n}$, as $n \rightarrow \infty$. Therefore:

$$|C_s|(1-p)^{b(s)} \leq \frac{s!}{2ns^{s+1}} \text{ for large values of } n \quad (6.26)$$

which shows that:

$$\sum_{s=2}^{3^{0.44n}} \sum_{S \in C_s} (1-p)^{b(S)} = o(1) \quad (6.27)$$

Finally, it remains to prove $\log_3(A \frac{2ns^{s+1}}{s!}) \rightarrow -\infty$ as $n \rightarrow \infty$. One can verify this for $s \leq n$ and $s > n$. After multiplying both sides of inequality (6.24) with $\frac{2ns^{s+1}}{s!}$ and then getting \log_3 from both sides, one gets:

$$\log_3(|C_s|(1-p)^{b(s)} \frac{2ns^{s+1}}{s!}) \leq s(\log_3 2n + \log_3 s + an \log_3(1-p) - a(\log_3 s) \log_3(1-p)) + n \quad (6.28)$$

For $s \leq n$ the largest factor in equation (6.28) is $n(a(\log_3(1-p))s + 1)$ which is negative for $p = p_c$ and $2 \leq s \leq n$. Therefore, $\log_3(A \frac{2ns^{s+1}}{s!}) \rightarrow -\infty$ as $n \rightarrow \infty$ for $s \leq n$. For $s > n$ the largest factor in equation (6.28) is $s(an(\log_3(1-p)) + (\log_3 s)(1 - a \log_3(1-p)))$ which is negative if $s < 3^{\frac{-a \log_3(1-p)}{1-a \log_3(1-p)}n} \approx 3^{0.44n}$. Therefore, $\log_3(A \frac{2ns^{s+1}}{s!}) \rightarrow -\infty$ as $n \rightarrow \infty$ for $s > n$. ■

For $p \geq 0.67$:

Theorem 15. If $p \geq 0.67$, the probability that for some $S \in C_s, 2 \leq s \leq 3^n$, no edges of $P_{3,p}^n$ join S to $V(P_3^n) \setminus S$, as $n \rightarrow \infty$, tends to 0. Hence, the probability that $P_{3,p}^n$ is connected tends to 1, as $n \rightarrow \infty$.

PROOF:

We know that:

$$|C_s| \leq \binom{3^n}{s} < \left(\frac{e3^n}{s}\right)^s \quad (6.29)$$

hence:

$$|C_s|(1-p)^{b(s)} \leq \binom{3^n}{s} < \left(\frac{e3^n}{s}\right)^s (1-p)^{\frac{e \ln 3}{3}s(n-\log_3 s)} \quad (6.30)$$

By using Mathematica ¹ one can show the right hand side of equation (6.30) tends to zero as $n \rightarrow \infty$. ■

¹Mathematica is a computational software program developed by Wolfram Research of Champaign, Illinois

CHAPTER 7

RELIABLE NETWORKS

In many engineering applications, it is of interest to construct a graph (network), with a specific number of edges and vertices, which is the most reliable one in that family of graph with n vertices and m edges. One of the measures of reliability is all-terminal reliability.

7.1 All terminal reliability

Definition 33. Random subgraph of a graph, Percolation on a graph: Random subgraph of $G(V_n, E_m)$ is the graph G_{p_n} which contains all vertices of G , and each edge of G independently with probability p_n . Doing percolation on a graph with the parameter p_n is the same as finding a random subgraph of a graph with the parameter p_n . p_n is called percolation parameter.

Definition 34. Uniformly optimally reliable graph (UOR): The UOR graph, if it exists, is the graph $G_n \in G(n, m)$ which maximizes the probability that G_n is connected after percolation with the parameter p_n for fixed n, m and all $p_n \in (0, 1)$.

Definition 35. Reliability polynomial: Let s_k be the number of spanning connected subgraphs of $G_n \in G(n, m)$ having exactly k edges. Let $R(G_n, p_n)$ be the probability that G_n is connected after percolation with the parameter p_n . One can formulate $R(G_n, p_n)$ as follows:

$$R(G_n, p_n) := \sum_{i=0}^m s_i p_n^i (1 - p_n)^{m-i} \quad (7.1)$$

$R(G_n, p_n)$ is called *reliability polynomial* of graph G_n or *all-terminal reliability*. In this definition, $s_i = 0$ for $i < n - 1$ and $s_m = 1$. Also, s_{m-1} is m -number of cuts in G_n , and s_{n-1} is the number of spanning tress of G_n .

The UOR graph, if it exists, is the graph $G_n \in G(n, m)$ which maximizes $R(G_n, p_n)$ for all $p_n \in (0, 1)$. From the definition of reliability polynomial one can see that the value of reliability polynomial depends on the structure of a graph as well as percolation value. Trivially, for fixed values of p_n there always exists an optimal solution for $R(G, p_n)$. In other words, for fixed values of p_n , there always exists a $G_n \in G(n, m)$ which maximizes $R(G_n, p_n)$. The following theorem and corollary can be helpful to find the UOR graph. This theorem and corollary are extracted from [17].

Theorem 16. Let G and H be two undirected simple graphs both having n nodes and m edges and $s_k(G), s_k(H)$ denote the number of spanning connected subgraphs of G and H , respectively, with exactly k edges [17].

1. If there exists an integer $0 \leq k \leq m-1$ such that $s_i(G) = s_i(H)$ for $i = 0, 1, \dots, k$ and $s_{k+1}(G) > s_{k+1}(H)$, then there exists a $\rho > 0$ such that for all $0 < p < \rho$ we have $R(G, p) > R(H, p)$.
2. If there exists an integer $0 \leq k \leq m$ such that $s_i(G) = s_i(H)$ for $i = m, m-1, \dots, m-k$ and $s_{m-k-1}(G) > s_{m-k-1}(H)$, then there exists a $\rho < 1$ such that for all $\rho < p < 1$ we have $R(G, p) > R(H, p)$.

Corollary 3. If G is UOR, then [17]:

1. G has the maximum number of spanning trees among all simple graphs having n nodes and m edges, and
2. G is $max - \lambda$, i.e. has the maximum possible value of λ among all simple graphs having n nodes and m edges, namely $\lambda(G) = \lfloor 2m/n \rfloor$, and the minimum number of cutsets of size λ among all such $max - \lambda$ graphs.

where $\lambda(G)$ is the edge connectivity of G , i.e., the minimum number of edges whose removal will disconnect G .

Important coefficients for large n : Let n be large and m sufficiently larger than n . When p_n is close to 0 then s_{n-1} , the number of spanning trees of G_n , has the most significant contribution in $R(G_n, p_n)$ since $(1 - p_n)$ is almost 1 and p_n^{n-1} is much larger than p_n^m . Similarly, when p_n is close to 1, then s_{m-1} , m -number of cuts, has the most significant contribution in $R(G_n, p_n)$.

Laplacian

The *Laplacian matrix* of a graph is described briefly in chapter 1. Here, we present a few results on Laplacian. The author calculated the algebraic connectivity for all graphs with $n = 5, 6, 7$ and $n-1 < m < \binom{n}{2}$ and could not find a direct relation between algebraic connectivity and all-terminal reliability. There is a room for further work in this part.

The Laplacian matrix is $L := (l_{i,j})_{n \times n}$ where $l := \begin{cases} \deg(v_i), & \text{if } i = j; \\ -1, & \text{if } i \neq j \text{ and } v_i \text{ adjacent to } v_j; \\ 0, & \text{o.w.} \end{cases}$.

Arrange the eigenvalues of L as: $\lambda_1(L) \leq \lambda_2(L) \leq \dots \leq \lambda_n(L)$. This set of λ_i 's are called the spectrum of L and $\lambda_2(L)$ is called the algebraic connectivity of a graph. The following interesting lemma sheds light on some applications of Laplacian matrix (this lemma is extracted from [24]):

Lemma 5. For a graph G on n vertices, we have [24]

(i):

$$\sum_i \lambda_i \leq n$$

with equality holding if and only if G has no isolated vertices.

(ii): For $n \geq 2$

$$\lambda_1 \leq \frac{n}{n-1}$$

with equality holding if and only if G is the complete graph on n vertices. Also, for a graph G without isolated vertices, we have:

$$\lambda_{n-1} \geq \frac{n}{n-1}.$$

n=5	s_{m-1}	s_{m-2}	s_{n-1}
m=5	5					
m=6	6	12				
m=7	7	20	24			
m=8	8	28	52	45		
m=9	9	36	82	111	75	
m=10	10	45	120	205	222	125

Table 7.1: The coefficients of the reliability polynomials of the UOR graph for $n = 6$ and $n - 1 < m < \binom{n}{2}$.

(iii:) For a graph which is not a complete graph, we have

$$\lambda_1 \leq 1.$$

(iv:) If G is connected, then $\lambda_1 > 0$. If $\lambda_i = 0$ and $\lambda_{i+1} \neq 0$, then G has exactly $i + 1$ connected components.

(v:) For all $i \leq n - 1$, we have:

$$\lambda_i \leq 2$$

with

$$\lambda_{n-1} = 2$$

if and only if a connected component of G is bipartite and nontrivial.

(vi:) The spectrum of a graph is the union of the spectrum of its connected components.

Reliability for $n = 5, 6, 7$

The author calculated $R(G, p)$ for $n = 5, 6, 7$ and $n - 1 < m < \binom{n}{2}$. The coefficients of the reliability polynomials are presented in tables 7.1, 7.2 and 7.3. As table 7.1 illustrates, for $n = 5$ there is always a UOR graph. For $n = 6, 7$ there are always a UOR graph except two cases. If, $(n, m) = (6, 11)$, the optimal solution for $R(G_n, p)$ depends on the value of p . For $p < 0.29$ (approximately 0.29) graph 7.1(a) is optimal while for $p > 0.29$ graph 7.1(b) optimal. Also, if $(n, m) = (7, 15)$, the optimal solution for $R(G_n, p)$ depends on the value of p . For $p < 0.81$ (approximately 0.81) graph 7.2(a) is optimal while for $p > 0.81$ graph 7.2(b) is optimal. From these observation one can conclude that the UOR does not exist for all values of n and m .

Reliability for $m = n - 1, n, n + 1, n + 2, n + 3$

For $m = n - 1, n, n + 1, n + 2$ there always exists a UOR graph. For $m = n - 1$, any tree is the UOR graph. For $m = n$, C_n , single cycle with n vertices, is the UOR graph. The first non-trivial case is $m = n + 1$, which is solved by F. Boesch [21], [20]. The UOR graph in this case is: for $n \geq 5$, start with a multigraph with 2 vertices and 3 edges. Then add total of $n - 2$ vertices of degree 2 in each lines of the graph so that the number of vertices in each line differs by at most one [20]. For $m = n + 2$ the problem is also solved by F. Boesch. The UOR graph in this case is: start with K_4 , then add total of $n - 2$ vertices of degree 2 in each lines of the graph so that the number of vertices in each line differs by at most one [20]. For $m = n + 3$, the UOR graph is found by G. Wang [22]. The UOR graph in this case is: start with $K_{3,3}$, a complete bipartite graph with 3 vertices in each part, and then add the remanning vertices as before.

n=6	s_{m-1}	s_{m-2}	s_{n-1}
m=6	6									
m=7	7	16								
m=8	8	26	36							
m=9	9	36	78	81						
m=10	10	45	116	177	135					
$m_a=11$	11	55	163	309	368	225				
$m_b=11$	11	55	163	310	370	224				
m=12	12	66	220	489	744	740	384			
m=13	13	78	286	771	1249	1552	1292	576		
m=14	14	91	364	999	1978	2877	3040	2196	864	
m=15	15	105	455	1365	2997	4945	6165	5700	3660	1296

Table 7.2: The coefficients of the reliability polynomials of the UOR graph for $n = 6$ and $n - 1 < m < \binom{n}{2}$. For $m = 11$, and $p < 0.29$ (approximately 0.29) the row with m_a is the UOR graph and for $p > 0.29$ the row with m_b is the UOR graph.

n=7	s_{m-1}	s_{m-2}	s_{n-1}
m=7	7													
m=8	8	21												
m=9	9	33	51											
m=10	10	44	104	117										
m=11	11	55	159	273	231									
m=12	12	66	216	456	612	432								
m=13	13	78	284	690	1146	1248								
m=14	14	91	364	994	1932	2668	720							
$m_a=15$	15	105	455	1360	2946	4704	5464	4320	1840					
$m_b=15$	15	105	455	1360	2946	4705	5465	4305	1805					
m=16	16	120	560	1817	4328	7766	10548	10628	7396	2800				
m=17	17	136	680	2379	6169	1226	18762	22226	19808	12320	4200			
m=18	18	153	816	3060	8562	18485	31344	41964	44000	35094	19716	6125		
m=19	19	171	969	3876	11624	27073	49985	73888	87468	81976	58958	30109	8575	
m=20	20	190	1140	4845	15502	38725	77240	124605	163400	173646	147500	96915	45530	12005
m=21	21	210	1330	5985	20349	54257	116175	202755	290745	343140	331506	258125	156555	68295
														16807

Table 7.3: The coefficients of the reliability polynomials of the UOR graph for $n = 7$ and $n - 1 < m < \binom{n}{2}$. For $m = 15$, and $p < 0.81$ (approximately 0.81) the row with m_a is the UOR graph and for $p > 0.81$ the row with m_b is the UOR graph.

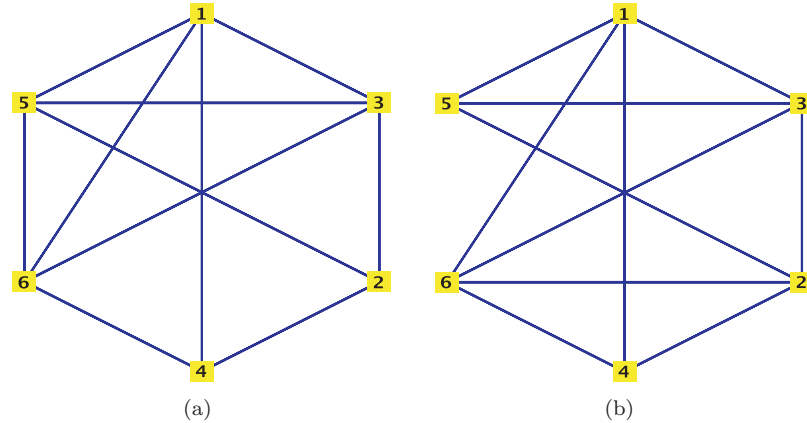


Figure 7.1: $(n, m) = (6, 11)$, For $p < 0.29$ (approximately 0.29) graph (a) is optimal while for $p > 0.29$ graph (b) is optimal.

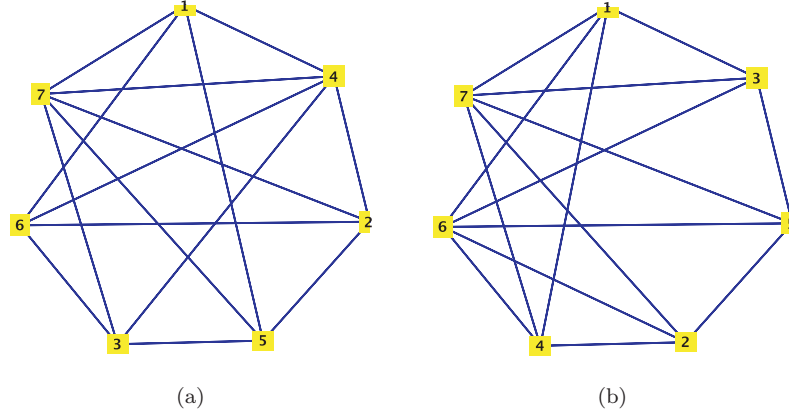


Figure 7.2: $(n, m) = (7, 15)$, For $p < 0.81$ (approximately 0.81) graph (a) is optimal while for $p > 0.81$ graph (b) is optimal.

Family of counterexamples

Kelmans [18] and Myrvold et al. [19] found infinite families of counter examples which the UOR graph does not exist. As an example, for n even and $n \geq 6$ and $m = n(n-1)/2 - (n+2)/2$, or for n odd and $n \geq 7$ and $m = n(n-1)/2 - (n+5)/2$ there always exists a graph in which it maximizes $R(G, p_n)$ for p close to 1, but do not have the maximum number of spanning trees. Therefore, from theorem 16 and corollary 3 the UOR graph does not exist for these families.

7.2 Random accessibility

M. Ebneshahrashoob, T. Gao and M. Sobel introduced the concept of random accessibility for simple graphs [23]. They believe that finding the UOR graph is related to the concept of random accessibility. In random accessibility, the goal is to find the expectation and the variance of the number of transitions X_j needed to visit j new vertices in $G \in G(V_n, E_m)$. In this approach the starting point is not considered as a new vertex. The result of the expectation and the variance can depend on starting point. Hence, one should change the starting point depending on the degree of it as a weighing factor. If one considers a graph with enough symmetry, the result does not depend on starting point. From analyzing numerical results, they make the following interesting conjecture:

- If the family of graphs contains both regular and non-regular graphs, then the UOR graph is among the regular graphs. Also, the expectation for random accessibility of graphs is equal or greater than the corresponding result of the UOR graph for each value of j close to $m-1$ (with the same ordering of the graph as for all-terminal reliability).

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